

Research paper

Regularity and stability analysis of stochastic space-time
fractional diffusion equations with multiplicative fractional
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ABSTRACT

This work investigates the regularity and stability of solutions to a class of stochastic space-time fractional diffusion with multiplicative fractional Brownian noise. The model involves both a time-fractional derivative and a space-fractional Laplacian, making it suitable for characterizing stochastic dynamical systems with memory effects and anomalous diffusion features. For the driving term, fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$, we first prove the existence and uniqueness of a mild solution under suitable Lipschitz and growth conditions, employing the theory of space-time fractional solution operators and regularity estimates. Moreover, the temporal and spatial regularity of the solution is also analyzed. The main contribution of this work is to establish the stability. It's proved, by deriving key prior estimates for the stochastic convolution term and utilizing the newly established exponential decay property of the fractional solution operator (Lemma 5) and a novel generalized Gronwall inequality (Lemma 6), that the solution is exponentially asymptotically stable in the p th moment sense ($p \geq 2$). This result demonstrates the long-term robustness of the system's dynamics to initial perturbations in a probabilistic sense. Compared to integer-order stochastic partial differential equations, the fractional solution operator lacks the semigroup property, and the covariance of fBm possesses singularities when $H \in (0, \frac{1}{2})$. This work innovatively overcomes these difficulties using new methods and results, thereby extending the existing stability theory for fractional stochastic partial differential equations [1]. Finally, numerical simulations confirm the theoretical findings.

1. Introduction

Stochastic partial differential equations (SPDEs) serve as fundamental mathematical tools for describing complex dynamical systems influenced by random factors, with wide applications in fluid dynamics, financial mathematics, and biological population dynamics, among others [2]. Traditional research has predominantly focused on integer-order diffusion models excited by standard Brownian motion (Bm). The theories concerning the regularity, well-posedness and stability of solutions for such equations are well-established (see the classical monograph [2]), including foundational work on moment decay and stability [3,4]. The analysis of solution regularity, especially Hölder continuity, has also been deeply studied in various contexts [5,6].

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In recent years, fractional calculus has garnered significant attention due to its ability to more accurately model anomalous diffusion processes characterized by memory effects (non-Markovianity) and spatial non-locality [7]. Consequently, the study of stochastic fractional partial differential equations (SFPDEs) has emerged, and significant progress has been made in numerical methods and unified convergence analysis [8–10]. As an important extension of standard Brownian motion, Fractional Brownian motion (fBm) has become a crucial noise model for its ability to capture the widespread long-range dependence (when $H \in (\frac{1}{2}, 1)$) or anti-persistence (when $H \in (0, \frac{1}{2})$) in real-world systems [11,12]. Unlike standard Brownian motion, whose increments are independent and whose sample paths are nowhere differentiable, fBm exhibits correlated increments, and the Hölder regularity of its sample paths is directly governed by the Hurst index H . This provides a more flexible framework for modeling physical processes with memory or rough characteristics, such as turbulence and financial fluctuations [13]. Therefore, the study of SPDEs forced by fBm has emerged, and significant progress has been made in numerical methods and unified convergence analysis [8–10].

Regarding stability, existing results are primarily concentrated on two types of models: systems driven by additive fBm, where stability analysis can rely on mean-square estimates of stochastic convolutions [14], and integer-order equations driven by multiplicative standard Bm, where the study of p th moment exponential stability typically depends on Itô's formula, Gronwall inequalities, and operator semigroup theory [15,16]. However, the stability analysis for spatio-temporal fractional diffusion equations driven by multiplicative fBm presents significant challenges and remains relatively scarce [17], especially for the Hurst index $H \in (0, \frac{1}{2})$. Most existing work focuses on the case $H \in (\frac{1}{2}, 1)$ (e.g., [1,18]), primarily due to two reasons. First, compared to standard Bm, the properties of fBm and the associated integration theory differ substantially: for $H \in (\frac{1}{2}, 1)$, fBm shows long-range dependence and relatively “smooth” paths, allowing its stochastic integral to be defined using Young integrals or pathwise Riemann-Stieltjes integrals, for which analytical tools are more mature [12]. In contrast, for $H \in (0, \frac{1}{2})$, the process exhibits anti-persistence and extremely irregular sample paths, rendering classical stochastic integration theories inapplicable and necessitating more complex techniques from Malliavin calculus or divergence-type integral theory [19,20], which greatly increases the difficulty of analysis. Second, the solution operator $\{\mathcal{R}_{\alpha,s}(t)\}_{t \geq 0}$ corresponding to the spatio-temporal fractional equation is no longer a semigroup, losing the favorable properties such as exponential decay and regularity transmission inherent to semigroups. This prevents the direct application of stability proof methods established within the classical semigroup framework, requiring the development of entirely new analytical techniques.

This paper aims to address this research gap by investigating the regularity and p th moment exponential stability of solutions to stochastic spatio-temporal fractional diffusion equations driven by multiplicative fBm with $H \in (0, 1)$, particularly the challenging case $H \in (0, \frac{1}{2})$. The main contributions are as follows. First, to the best of the authors' knowledge, this work is among the first to extend the regularity and stability theory for stochastic equations driven by multiplicative fractional Brownian motion to the setting of space-time fractional diffusion equations, and it permits the Hurst index to cover the entire interval $H \in (0, 1)$. Second, since the solution operator $\{\mathcal{R}_{\alpha,s}(t)\}_{t \geq 0}$ lacks the semigroup property, a key exponential decay estimate is established (Lemma 5) via Laplace transform, spectral theory and complex analysis. This estimate underpins all subsequent analysis. Third, estimating stochastic integrals for rough fBm is non-trivial. The Malliavin divergence integral is employed. Meanwhile imposing growth and Lipschitz conditions on the Malliavin derivative of the coefficient (Assumptions (2.20) and (2.21)) together with a priori convolution estimates and time regularity of the solution operator, overcomes the covariance singularity. It should be noted that when $H < 1/2$, inspired by [21], an attempt has been made to relax the conditions (2.20) and (2.21) to an integrated form, under which existence, uniqueness and space-time regularity still follow (Remark 2). Fourth, classical Gronwall inequalities are insufficient for the stability proof, due to the presence of multiple weakly singular convolution terms. To overcome this difficulty, and inspired by the three-term inequality in [5], a new generalized Gronwall-type inequality (Lemma 6) is constructed, which serves as the core tool for deriving the exponential decay rate of the stability result.

The structure of this work is outlined as follows. Section 2 recalls the basic concepts, properties, and necessary assumptions required for the subsequent analysis, including fractional calculus, fBm, divergence integrals, and relevant function spaces. Section 3 gives the definition of the solution and demonstrates its existence and uniqueness. Furthermore, through detailed a priori estimates, the spatio-temporal regularity properties of the solution are studied. Section 4 constitutes the core of this paper. First, we establish a new decay lemma for the solution operator (Lemma 5) and a crucial generalized Gronwall inequality (Lemma 6). Subsequently, by comprehensively applying stochastic analysis techniques and the newly established inequalities, we rigorously prove that the mild solution of the system is exponentially asymptotically stable with respect to the p th moment. Section 5 is dedicated to validating the theoretical results through numerical simulation. We design a finite difference scheme to discretize the model and conduct numerical experiments to visually demonstrate the dynamic behavior of the system's solution trajectories and their trend toward exponential stability under different Hurst indices H and fractional parameters α, s . Finally, Section 6, the conclusion and outlook, summarizes the main findings of this paper.

2. Preliminaries

2.1. Model introduction

This work investigates the asymptotic properties of solutions for a stochastic dynamical system involving a time-fractional derivative and perturbed by multiplicative fractional Brownian noise. The model under consideration reads

$$dy(t) + {}_0\partial_t^{1-\alpha} \mathbb{A}^s y(t) dt = \mathbf{h}(y(t)) dt + \mathbf{g}(y(t)) dB^H(t), \quad t \in [0, T], \quad y(0) = y_0 \quad (2.1)$$

with $T > 0$ being a given constant. Let $U = L^2(D)$ with inner product $\langle \cdot, \cdot \rangle_U$. For $s \in (0, 1)$, $\mathbb{A}^s$ denotes the fractional Laplacian, defined by:

$$\mathbb{A}^s y = \sum_{k=1}^{\infty} \lambda_k^s \langle y, \psi_k \rangle_U \psi_k.$$

Here $\mathbb{A} = -\Delta$ is the Laplace operator with homogeneous Dirichlet boundary conditions, whose eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ are non-decreasing, and the corresponding orthonormal eigenfunctions in U are $\{\psi_k\}_{k=1}^{\infty}$. Introduce the fractional-order space induced by $\mathbb{A}^s$:

$$U^s := \text{dom}(\mathbb{A}^s),$$

and its norm $\|\cdot\|_{U^s}$ is defined as:

$$\|v\|_{U^s}^2 := \|\mathbb{A}^{\frac{s}{2}} v\|^2 = \sum_{k=1}^{\infty} \lambda_k^s \langle v, \psi_k \rangle_U^2.$$

For any $s > 0$, U^s equipped with $\|\cdot\|_{U^s}$ forms a Banach space.

Definition 1. ([22]) ${}_0\partial_t^{1-\alpha}$ with $\alpha \in (0, 1)$ denotes the Riemann-Liouville fractional derivative, whose definition can be given as follows:

$${}_0\partial_t^{1-\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-u)^{\alpha-1} y(u) du. \quad (2.2)$$

2.2. Stochastic calculus theory of fBm

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ be a complete filtered probability space satisfying the usual conditions (right-continuity of the filtration and $\mathcal{F}_0$ containing all $\mathbb{P}$ -null sets). Let V be a separable Hilbert spaces with inner product $\langle \cdot, \cdot \rangle_V$ and a corresponding norm $\|\cdot\|_V$. The collection of bounded linear operators from U to V is denoted by $\mathcal{L}(U; V)$. For notational simplicity, let $\mathcal{L}(U) = \mathcal{L}(U; U)$. The spaces of nuclear and Hilbert-Schmidt operators on U are denoted by $\mathcal{L}_1(U)$ and $\mathcal{L}_2(U)$, respectively, where $\mathcal{L}_2(U)$ is equipped with the inner product

$$\langle S, T \rangle_{\text{HS}} = \sum_{k=1}^{\infty} \langle S \psi_k, T \psi_k \rangle_U.$$

The Hilbert-Schmidt norm is written as $\|\cdot\|_{\text{HS}}$. Let Q be a nonnegative, self-adjoint, trace-class operator (the covariance operator) with eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and eigenfunctions $\{\psi_k\}_{k=1}^{\infty}$. Define $U_0 := Q^{1/2}(U)$. The Banach space $\mathcal{L}_2^0$ is then introduced, consisting of all Hilbert-Schmidt operators from U_0 into U , equipped with the inner product

$$\langle \Phi_1, \Phi_2 \rangle_{\mathcal{L}_2^0} = \sum_{k=1}^{\infty} \langle \Phi_1 Q^{1/2} \psi_k, \Phi_2 Q^{1/2} \psi_k \rangle_U.$$

One easily verifies the following relation between the two norms:

$$\|\mathcal{T}\|_{\mathcal{L}_2^0} = \|\mathcal{T} Q^{1/2}\|_{\text{HS}}.$$

Definition 2. A one-dimensional fractional Brownian motion $\{\beta^H(t)\}_{t \geq 0}$ with Hurst index $H \in (0, 1)$ is defined as a centered Gaussian process satisfying $\beta^H(0) = 0$ and the covariance

$$\mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad s, t \geq 0. \quad (2.3)$$

When $H = 1/2$, this reduces to the standard Brownian motion.

Let $\{\beta_k^H(t)\}_{k=1}^{\infty}$ be a family of independent one-dimensional fBms with common Hurst index $H \in (0, 1)$, all adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. The U -valued fBm $B^H(t)$ is then constructed via the following series expansion:

$$B^H(t) := \sum_{k=1}^{\infty} \sqrt{\lambda_k} \psi_k \beta_k^H(t), \quad t \geq 0. \quad (2.4)$$

A one-dimensional fBm $\beta^H(t) = \int_0^t \mathbf{K}_H(t, s) d\beta(s)$ can also be represented as a stochastic-integral, and the kernel $\mathbf{K}_H$ is piecewise defined as

$$\mathbf{K}_H(t, s) = \begin{cases} c_{H,1} \left[\left(\frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right], \\ \quad H \in \left(0, \frac{1}{2} \right), \\ c_{H,2} \int_s^t \left(\frac{u}{s} \right)^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du, \quad H \in \left(\frac{1}{2}, 1 \right), \end{cases} \quad (2.5)$$

where the positive constants $c_{H,1}$ and $c_{H,2}$ are given in terms of the beta function $B(\cdot, \cdot)$ by

$$c_{H,1} = \left(\frac{2H}{(1-2H)B(1-2H, H + \frac{1}{2})} \right)^{\frac{1}{2}}, \quad c_{H,2} = \left(\frac{H(2H-1)}{B(2-2H, H - \frac{1}{2})} \right)^{\frac{1}{2}}.$$

Now, define the linear operator $\Gamma_{H,t}^*$ from the ordinary function space to $L^2([0, T])$ as:

$$(\Gamma_{H,t}^* \eta)(u) := \mathbf{K}_H(t, u) \eta(u) + \int_u^t (\eta(r) - \eta(u)) \frac{\partial \mathbf{K}_H(r, u)}{\partial r} dr. \quad (2.6)$$

One can compute the partial derivative of the kernel $\mathbf{K}_H(r, u)$ in r as:

$$\frac{\partial \mathbf{K}_H(r, u)}{\partial r} = \mathbf{K}_H\left(\frac{r}{u}\right)^{H-\frac{1}{2}} (r-u)^{H-\frac{3}{2}}. \quad (2.7)$$

When the Hurst index $H \in (\frac{1}{2}, 1)$, the operator $\Gamma_{H,t}^*$ reduces to:

$$(\Gamma_{H,t}^* \varphi)(u) := \int_u^t \varphi(r) \frac{\partial \mathbf{K}_H(r, u)}{\partial r} dr. \quad (2.8)$$

Definition 3. ([12,23,24]) Suppose that

$$\mathbf{G} = g\left(\int_0^T h_1(r) dB^H(r), \dots, \int_0^T h_n(r) dB^H(r)\right),$$

where $h_i = \Gamma_{H,t}^* e_i$, and e_i is an orthogonal basis in $L^2([0, T])$, $i = 1, 2, \dots$. For the random variable $\mathbf{G}$, its Malliavin derivative $\mathcal{D}_s^H \mathbf{G}$ is defined as

$$\mathcal{D}_s^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \left(\int_0^T h_1(r) dB^H(r), \dots, \int_0^T h_n(r) dB^H(r) \right) h_i(s), \quad s \in [0, T].$$

To handle the multiplicative noise term in Eq. (2.1), especially for the rough case $H \in (0, \frac{1}{2})$, the stochastic integral must be defined within the framework of the Malliavin divergence integral. This subsection presents the requisite definitions, existence theorem, and a crucial moment estimate, which form the foundation for the subsequent analysis. The full theory can be found in [12,21,27].

A suitable space of stochastic processes for which the divergence integral with respect to $B^H(r)$ is well-defined is first introduced. Let $\mathbb{D}_s^H$ be defined as $\mathbb{D}_s^H = \Gamma_{H,t}^* \Gamma_{H,t}^* \mathcal{D}_s^H$.

Definition 4. Denote by $\mathbb{L}_H^{1,2}$ the set of all measurable $\mathcal{L}_2^0$ -valued stochastic processes $Y : [0, T] \times \Omega \rightarrow \mathcal{L}_2^0$ satisfying:

1. $\mathbb{E} \left[\int_0^T \|Y(u)\|_{\mathcal{L}_2^0}^2 du \right] < \infty$;
2. For almost all $u \in [0, T]$, $Y(u)$ belongs to the Malliavin differentiable space $\mathbb{D}^{1,2}(\mathcal{L}_2^0)$, and its Malliavin derivative $\mathbb{D}_s^H Y(u)$ satisfies:

$$\mathbb{E} \left[\int_0^T \int_0^T \|\mathbb{D}_s^H Y(u)\|_{\mathcal{L}_2^0}^2 ds du \right] < \infty.$$

For a predictable process $Y \in \mathbb{L}_H^{1,2}$, its divergence integral (or Skorokhod integral) $\int_0^T Y(u) dB^H(u)$ with respect to fBm is well-defined. The following proposition states a key sufficient condition for its existence and provides its second-moment formula, which is the starting point for a priori estimates.

Let $Y \in \mathbb{L}_H^{1,2}$ be a predictable process that additionally satisfies the following cross-expectation integrability condition:

$$\mathbb{E} \left[\int_0^T \int_0^T \left\langle (\Gamma_{H,T}^*)^{\otimes 2} \mathbb{D}_s^H Y(u), (\Gamma_{H,T}^*)^{\otimes 2} \mathbb{D}_u^H Y(s) \right\rangle_{\mathcal{L}_2^0} ds du \right] < \infty, \quad (2.9)$$

where $\Gamma_{H,T}^*$ is the linear operator defined in (2.6) and (2.8). Then the divergence integral exists, belongs to $L^2(\Omega; U^s)$, and satisfies:

$$\mathbb{E} \left[\int_0^T Y(u) dB^H(u) \right] = 0, \quad (2.10)$$

$$\mathbb{E} \left\| \int_0^T Y(u) dB^H(u) \right\|_{U^s}^2 = \mathbb{E} \|Y\|_{\Gamma_H}^2 + \mathbb{E} \left[\int_0^T \int_0^T \left\langle (\Gamma_{H,T}^*)^{\otimes 2} \mathbb{D}_s^H Y(u), (\Gamma_{H,T}^*)^{\otimes 2} \mathbb{D}_u^H Y(s) \right\rangle_{\mathcal{L}_2^0} ds du \right], \quad (2.11)$$

where $\|Y\|_{\Gamma_H}^2 := \int_0^T \|\Gamma_{H,t}^* Y(u)\|_{\mathcal{L}_2^0}^2 dt$.

Remark 1. Eq. (2.11) is the cornerstone for proving the second-moment estimates of the solution (e.g., in Theorem 2). For the specific integrand $Y(t) = \mathcal{R}_{\alpha,s}(t-u)g(y(u))$, this formula leads directly to the estimation structure seen in the proof of Lemma 2.

The following generalized moment estimate is crucial for establishing p th moment regularity and stability; it can be seen as a fractional counterpart of the Burkholder-Davis-Gundy inequality for standard Brownian motion.

Lemma 1. Let $p \geq 2$ and $Y \in \mathbb{L}_H^{1,2}$ be a predictable. Then one can find a positive constant C_p (which depends on T , H , and p) such that

$$\mathbb{E} \left\| \int_0^t Y(u) dB^H(u) \right\|_{U^s}^p \leq C_p \left\{ \mathbb{E} \left(\int_0^t \| \Gamma_{H,t}^* Y(u) \|_{\mathcal{L}_2^0}^2 du \right)^{\frac{p}{2}} + \mathbb{E} \left(\int_0^t \int_0^t \| \mathbb{D}_r^H Y(u) \|_{\mathcal{L}_2^0}^2 dr du \right)^{\frac{p}{2}} \right\} \quad (2.12)$$

holds for all $t \in [0, T]$.

2.3. Properties of the fractional solution operator

Firstly, the fractional solution operator $\mathcal{R}_{\alpha,s}(t)$ defined as:

$$\mathcal{R}_{\alpha,s}(t) := \frac{1}{2\pi i} \int_{\Gamma_3} e^{\zeta t} \zeta^{\alpha-1} (\zeta^\alpha + \mathbb{A}^s)^{-1} d\zeta, \quad (2.13)$$

where Γ_3 is a contour to be specified.

Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ be sets. Let $m > 0$, $\frac{\pi}{2} < \theta < \pi$. Define:

$$\Gamma_1 = \{\zeta \in \mathbb{C} : \zeta \neq 0, |\arg(\zeta)| \leq \theta\},$$

$$\Gamma_2 = \{\zeta \in \mathbb{C} : |\zeta| \geq m, |\arg(\zeta)| \leq \theta\},$$

$$\Gamma_3 = \{re^{i\theta} : r \geq m\} \cup \{me^{i\varphi} : |\varphi| \leq \theta\} \cup \{re^{-i\theta} : r \geq m\}.$$

Here i is the imaginary unit belonging to the complex plane $\mathbb{C}$.

The Laplace transform of $\mathcal{R}_{\alpha,s}(t)$ is given by:

$$\widetilde{\mathcal{R}}_{\alpha,s}(\zeta) = \zeta^{\alpha-1} (\zeta^\alpha + \mathbb{A}^s)^{-1}. \quad (2.14)$$

Using the inverse Laplace transform and interpolation properties, we obtain the key regularity estimate:

$$\|\mathbb{A}^{s\beta} \mathcal{R}_{\alpha,s}(t)\| \leq Ct^{-\beta\alpha}, \quad \text{where } \beta \in [0, 1]. \quad (2.15)$$

2.4. Assumptions regarding coefficients $\mathbf{h}$ and $\mathbf{g}$

Set $\mathbb{D}_s^H$ to denote the Malliavin derivative. Let $C > 0$ be a constant. The inequalities below hold for all $y, z \in U^s$:

$$\|\mathbf{h}(y)\|_{U^s} \leq C(1 + \|y\|_{U^s}), \quad (2.16)$$

$$\|\mathbf{h}(y) - \mathbf{h}(z)\|_{U^s} \leq C\|y - z\|_{U^s}, \quad (2.17)$$

$$\|\mathbb{A}^{\frac{s}{2}} \mathbf{g}(y)\|_{\mathcal{L}_2^0} = \|\mathbf{g}(y)\|_{\mathcal{L}_{2,s}^0} \leq C(1 + \|y\|_{U^s}), \quad (2.18)$$

$$\|\mathbb{A}^{\frac{s}{2}} (\mathbf{g}(y) - \mathbf{g}(z))\|_{\mathcal{L}_2^0} \leq C\|y - z\|_{U^s}, \quad (2.19)$$

$$\|\mathbb{D}_r^H \mathbb{A}^{\frac{s}{2}} \mathbf{g}(y)\|_{\mathcal{L}_2^0} \leq C(1 + \|y\|_{U^s}), \quad (2.20)$$

$$\|\mathbb{D}_r^H \mathbb{A}^{\frac{s}{2}} (\mathbf{g}(y) - \mathbf{g}(z))\|_{\mathcal{L}_2^0} \leq C\|y - z\|_{U^s}. \quad (2.21)$$

Remark 2. Assumptions (2.20) and (2.21) impose growth and Lipschitz conditions on the Malliavin derivative of the coefficient $\mathbf{g}$. When $H \in (0, \frac{1}{2})$, the linear example does not satisfy the assumptions (2.20) and (2.21). Based on [21], we find that one may attempt to relax conditions (2.20) and (2.21) to integral estimates. Specifically, taking $\mathbf{g}(y) = \lambda y$ with a constant λ , one may use the definition of the Malliavin derivative to derive $\int_0^T \|\mathbb{D}_r^H \mathbf{g}(y(t))\|_{\mathcal{L}_2^0}^2 ds \leq C\|y(t)\|_{H^s}^2$. Building upon this estimate, the following integrated bound involving the fractional operator can be obtained:

$$\int_0^T \|\mathbb{D}_r^H \mathbb{A}^{s/2} \mathbf{g}(y(t))\|_{\mathcal{L}_2^0}^2 ds \leq C\|y(t)\|_{U^s}^2, \quad (2.22)$$

$$\int_0^T \|\mathbb{D}_s^H \mathbb{A}^{s/2} (\mathbf{g}(y(t)) - \mathbf{g}(z(t)))\|_{\mathcal{L}_2^0}^2 ds \leq C\|y(t) - z(t)\|_{U^s}^2. \quad (2.23)$$

Under this integrated conditions (2.22) and (2.23), the existence, uniqueness (Theorem 1), and space-time regularity (Theorems 2 and 3) of the solution can still be derived. The stability result (Theorem 4) at present still relies on (2.20) and (2.21), which limits the wide applicability of the results. It is hoped that further investigation can be carried out in the future.

3. Proofs of existence, uniqueness, and regularity

3.1. Mild solution

To start, consider the concept of a mild solution to Eq. (2.1).

Definition 5. A predictable U^s -valued stochastic process $\{y(t)\}_{t \in [0, T]}$ is said to be a mild solution to (2.1) if the solution operator Φ_{y_0} , defined by

$$\Phi_{y_0}(y)(t) := \mathcal{R}_{\alpha, s}(t)y_0 + \int_0^t \mathcal{R}_{\alpha, s}(t-u)\mathbf{h}(y(u)) du + \int_0^t \mathcal{R}_{\alpha, s}(t-u)\mathbf{g}(y(u)) dB^H(u), \quad (3.1)$$

is well-defined, and the process $y(t)$ satisfies the fixed-point relation

$$y(t) = \Phi_{y_0}(y)(t) \quad \mathbb{P}\text{-a.s.},$$

for almost every $t \in [0, T]$.

To analyze the well-posedness of the mild solution, a central task is to control the stochastic convolution term. We therefore define

$$\mathcal{B}_t := \int_0^t \mathcal{R}_{\alpha, s}(t-u)\mathbf{g}(y(u)) dB^H(u), \quad t \in [0, T]. \quad (3.2)$$

The estimation of $\mathcal{B}_t$ constitutes a key step in the forthcoming analysis. Throughout the rest of the paper, $C > 0$ denotes a constant that may vary from line to line and depends only on the parameters.

Lemma 2. Let $p \geq 2$, $\alpha < 2H$. Suppose (2.16) and (2.20) hold. Then a constant C satisfying the following is obtained:

$$\|\mathcal{B}_t\|_{L^p(\Omega; U^s)} \leq C \left(1 + \sup_{u \in [0, T]} \|y(u)\|_{L^p(\Omega; U^s)} \right) t^{\frac{2H-\alpha}{2}}.$$

Proof. Thanks to [21], combining (2.16) yields:

$$\begin{aligned} \|\mathcal{B}_t\|_{L^p(\Omega; U^s)} &\leq C \left\| \left(\int_0^t \left\| \Gamma_{H, t}^* \mathcal{R}_{\alpha, s}(t-u)\mathbf{g}(y(u)) \right\|_{\mathcal{L}_{2, s}^0}^2 du \right)^{\frac{1}{2}} \right\|_{L^p(\Omega; \mathbb{R})} \\ &\quad + C \left\| \left(\int_0^t \int_0^t \left\| \mathbb{D}_r^H \mathcal{R}_{\alpha, s}(t-u)\mathbf{g}(y(u)) \right\|_{\mathcal{L}_{2, s}^0}^2 dr du \right)^{\frac{1}{2}} \right\|_{L^p(\Omega; \mathbb{R})} \\ &\leq C \left(1 + \sup_{u \in [0, T]} \|y(u)\|_{L^p(\Omega; U^s)} \right) \left(\int_0^t \left\| \Gamma_{H, t}^* \mathcal{R}_{\alpha, s}(t-u) \right\|_{\mathcal{L}_{2, s}^0}^2 du \right)^{\frac{1}{2}} \\ &\quad + C \left\| \left(\int_0^t \int_0^t \left\| \mathbb{D}_r^H \mathcal{R}_{\alpha, s}(t-u)\mathbf{g}(y(u)) \right\|_{\mathcal{L}_{2, s}^0}^2 dr du \right)^{\frac{1}{2}} \right\|_{L^p(\Omega; \mathbb{R})} \\ &\leq C \left(1 + \sup_{u \in [0, T]} \|y(u)\|_{L^p(\Omega; U^s)} \right) I_1^{\frac{1}{2}} + I_2. \end{aligned}$$

Firstly, according to (2.15) and (2.20), it follows that :

$$\begin{aligned} I_2 &\leq C \left\| \left(\int_0^t \int_0^t \left\| \mathbb{D}_r^H \mathbf{g}(y(u)) \right\|_{\mathcal{L}_{2, s}^0}^2 \left\| \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha, s}(t-u) \right\|^2 dr du \right)^{\frac{1}{2}} \right\|_{L^p(\Omega; \mathbb{R})} \\ &\leq C \left(\sup_{u \in [0, T]} \|y(u)\|_{L^p(\Omega; U^s)} + 1 \right) \left(\int_0^t \int_0^t (t-u)^{-\alpha} dr du \right)^{\frac{1}{2}} \\ &\leq C \left(1 + \sup_{u \in [0, T]} \|y(u)\|_{L^p(\Omega; U^s)} \right) t^{1-\frac{\alpha}{2}} \\ &\leq C \left(1 + \sup_{u \in [0, T]} \|y(u)\|_{L^p(\Omega; U^s)} \right) t^{\frac{2H-\alpha}{2}}. \end{aligned} \quad (3.3)$$

Now, for $H \in (0, \frac{1}{2})$, (2.6) yields an estimate for I_1 :

$$\begin{aligned} I_1 &= C \int_0^t \left\| \Gamma_{H, t}^* \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha, s}(t-r) \right\|_{\mathcal{L}_{2, s}^0}^2 dr \\ &\leq C \int_0^t \left\| \mathbf{K}_H(t, r) \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha, s}(t-r) \right\|^2 dr \\ &\quad + C \int_0^t \left\| \int_r^t [\mathcal{R}_{\alpha, s}(t, u) - \mathcal{R}_{\alpha, s}(t, r)] \frac{\partial \mathbf{K}_H(u, r)}{\partial u} du \right\|^2 dr \\ &:= J_1 + J_2. \end{aligned} \quad (3.4)$$

Then J_1 is estimated. According to (2.5), the following estimate is obtained:

$$J_1 \leq \int_0^t \left\| \left(\frac{t}{r} \right)^{H-\frac{1}{2}} (t-r)^{H-\frac{1}{2}} \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha, s}(t-r) \right\|^2 dr$$

$$\begin{aligned}
& + C \int_0^t \left\| r^{\frac{1}{2}-H} \int_r^t u^{H-\frac{3}{2}} (u-r)^{H-\frac{1}{2}} du \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-r) \right\|^2 dr \\
& = J_{1,1} + J_{1,2}.
\end{aligned} \tag{3.5}$$

Assume $\alpha < 2H$:

$$\begin{aligned}
J_{1,1} & \leq \int_0^t \left\| \left(\frac{t}{r} \right)^{H-\frac{1}{2}} (t-r)^{H-\frac{1}{2}-\frac{\alpha}{2}} \right\|^2 dr \\
& \leq \int_0^t \left(\frac{t}{r} \right)^{2H-1} (t-r)^{2H-\alpha-1} dr \leq C t^{2H-\alpha}.
\end{aligned} \tag{3.6}$$

About the estimate of $J_{1,2}$, using the two-step integral substitution $u = r/\Theta$ and $r = w t$ gives:

$$\begin{aligned}
J_{1,2} & \leq C \int_0^t \left(r^{\frac{1}{2}-H} (t-r)^{-\frac{\alpha}{2}} \left[\int_r^t u^{H-\frac{3}{2}} (u-r)^{H-\frac{1}{2}} du \right]^2 \right) dr \\
& \leq C \int_0^t r^{1-2H} (t-r)^{-\alpha} \left(\int_r^t u^{H-\frac{3}{2}} (u-r)^{H-\frac{1}{2}} du \right)^2 dr \\
& \leq C \int_0^t r^{2H-1} (t-r)^{-\alpha} \left(\int_{\frac{r}{t}}^1 \Theta^{-2H} (1-\Theta)^{H-\frac{1}{2}} d\Theta \right)^2 dr \\
& \leq C \int_0^1 w^{2H-1} (1-w)^{-\alpha} \left(\int_w^1 \Theta^{-2H} (1-\Theta)^{H-\frac{1}{2}} d\Theta \right)^2 dw \cdot t^{2H-\alpha} \\
& \leq C t^{2H-\alpha}.
\end{aligned} \tag{3.7}$$

Hence $J_1 \leq J_{1,1} + J_{1,2} \leq C t^{2H-\alpha}$.

Now, using the inequality in [25], an estimate for J_2 is obtained as follows:

$$\left| \frac{e^{\zeta u} - 1}{u^\gamma} \right| < C |\zeta|^\gamma, \quad \gamma \in [0, 1].$$

Set $\gamma = \frac{1}{2} - H + \varepsilon$ with $\varepsilon > 0$ arbitrarily small. Gathering (2.13) and (2.15) gives:

$$\begin{aligned}
J_2 & = \int_0^t \left\| \int_r^t \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-u) - \mathcal{R}_{\alpha,s}(t-r)] C_H \left(\frac{u}{r} \right)^{H-\frac{1}{2}} (u-r)^{H-\frac{3}{2}} du \right\|^2 dr \\
& = C \int_0^t \left\| \int_r^t \left[\frac{1}{2\pi i} \int_{\Gamma_3} [e^{\zeta(t-u)} - e^{\zeta(t-r)}] \mathbb{A}^{\frac{s}{2}} \widetilde{\mathcal{R}}_{\alpha,s}(\zeta) d\zeta \right] \left(\frac{u}{r} \right)^{H-\frac{1}{2}} (u-r)^{H-\frac{3}{2}} du \right\|^2 dr \\
& \leq C \int_0^t \left\| \int_r^t \left[\frac{1}{2\pi i} \int_{\Gamma_3} |e^{\zeta(t-u)}| |1 - e^{\zeta(u-r)}| |\zeta|^{\frac{\alpha}{2}-1} |d\zeta| \right] \left(\frac{u}{r} \right)^{H-\frac{1}{2}} (u-r)^{H-\frac{3}{2}} du \right\|^2 dr \\
& \leq C \int_0^t \left\| \int_r^t \left[\frac{1}{2\pi i} \int_{\Gamma_3} |e^{\zeta(t-u)}| |\zeta|^{\gamma-1+\frac{\alpha}{2}} (u-r)^\gamma |d\zeta| \right] \left(\frac{u}{r} \right)^{H-\frac{1}{2}} (u-r)^{H-\frac{3}{2}} du \right\|^2 dr \\
& \leq C \int_0^t \left(\int_r^t (t-u)^{-\gamma-\frac{\alpha}{2}} (u-r)^{\gamma+H-\frac{3}{2}} du \right)^2 dr \\
& \leq C \int_0^t \left[(t-r)^{H-\frac{\alpha}{2}-\frac{1}{2}} \right]^2 dr \leq C t^{2H-\alpha}.
\end{aligned} \tag{3.8}$$

Therefore, when $H \in (0, \frac{1}{2})$, the following is obtained:

$$\|\mathcal{R}_t\|_{L^p(\Omega; U^s)} \leq (1 + \sup_{u \in [0, T]} \|y(u)\|_{L^p(\Omega; U^s)}) t^{\frac{2H-\alpha}{2}}.$$

In the case of $H \in (\frac{1}{2}, 1)$, (2.6), (2.8), and (2.15) together yield:

$$\begin{aligned}
I & = \int_0^t \left\| \Gamma_{H,t}^* \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-r) \right\|^2 dr \\
& = \int_0^t \left\| \int_r^t \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-r) \left(\frac{u}{r} \right)^{H-\frac{1}{2}} (u-r)^{H-\frac{3}{2}} du \right\|^2 dr \\
& \leq C \int_0^t \left[\int_r^t (t-r)^{\frac{\alpha}{2}} \left(\frac{u}{r} \right)^{H-\frac{1}{2}} (u-r)^{H-\frac{3}{2}} du \right]^2 dr
\end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^t (t-r)^{-\alpha} \left(\frac{t}{r}\right)^{2H-1} \left(\int_r^t (u-r)^{H-\frac{3}{2}} du\right)^2 dr \\
&\leq C \int_0^t (t-r)^{2H-\alpha-1} \left(\frac{t}{r}\right)^{2H-1} dr \\
&\leq Ct^{2H-\alpha}.
\end{aligned} \tag{3.9}$$

That is, when $H \in (\frac{1}{2}, 1)$,

$$\|\mathcal{B}_t\|_{L^p(\Omega; U^s)} \leq \left(1 + \sup_{u \in [0, T]} \|y(u)\|_{L^p(\Omega; U^s)}\right) t^{\frac{2H-\alpha}{2}}. \tag{3.10}$$

□

3.2. Existence and uniqueness

Lemma 3. ([5]) Denote by Θ_s^p the set of all predictable U^s -valued processes $\{y(t) : t \in [0, T]\}$. For any $M \in (0, \infty)$, the subset

$$\Theta_s^p(M) = \{y(t) \in \Theta_s^p : \|y\|_{\Theta_s^p} \leq M\}$$

is a Banach space, where the norm is given by

$$\|y(t)\|_{\Theta_s^p} := \sup_{t \in [0, T]} \|y(t)\|_{L^p(\Omega; U^s)} < \infty$$

with

$$\|y(t)\|_{L^p(\Omega; U^s)} = (\mathbb{E} \|y(t)\|_s^p)^{\frac{1}{p}}.$$

Theorem 1. Assume that $y(t)$ and $z(t)$ are solutions of (2.1), and assumptions of Lemma 2 together with (2.16) hold. Choose a sufficiently small $\tau > 0$ and take $t \in [0, \tau]$. Then Eq. (2.1) has a unique global mild solution.

Proof. If $y(t) \in \Theta_s^p$, using (2.16) and Lemma 2:

$$\begin{aligned}
\|\Phi_{y_0}(y)(t)\|_{\Theta_s^p} &= \left\| \mathcal{R}_{\alpha, s}(t)y_0 + \int_0^t \mathcal{R}_{\alpha, s}(t-r)\mathbf{h}(y(r)) dr \right. \\
&\quad \left. + \int_0^t \mathcal{R}_{\alpha, s}(t-r)\mathbf{g}(y(r)) dB^H(r) \right\|_{\Theta_s^p} \\
&\leq \left\| \mathcal{R}_{\alpha, s}(t)y_0 \right\|_{\Theta_s^p} + \left\| \int_0^t \mathcal{R}_{\alpha, s}(t-r)\mathbf{h}(y(r)) dr \right\|_{\Theta_s^p} \\
&\quad + \left\| \int_0^t \mathcal{R}_{\alpha, s}(t-r)\mathbf{g}(y(r)) dB^H(r) \right\|_{\Theta_s^p} \\
&= \sup_{t \in [0, T]} \left\| \mathcal{R}_{\alpha, s}(t)y_0 \right\|_{L^p(\Omega; U^s)} \\
&\quad + \sup_{t \in [0, T]} \left\| \int_0^t \mathcal{R}_{\alpha, s}(t-r)\mathbf{h}(y(r)) dr \right\|_{L^p(\Omega; U^s)} \\
&\quad + \sup_{t \in [0, T]} \left\| \int_0^t \mathcal{R}_{\alpha, s}(t-r)\mathbf{g}(y(r)) dB^H(r) \right\|_{L^p(\Omega; U^s)} \\
&\leq C + \sup_{t \in [0, T]} \int_0^t \left\| \mathcal{R}_{\alpha, s}(t-r) \right\| \left\| \mathbf{h}(y(r)) \right\|_{L^p(\Omega; U^s)} dr \\
&\quad + \sup_{t \in [0, T]} \left\| \mathcal{B}_t \right\|_{L^p(\Omega; U^s)} \\
&\leq C + C \int_0^t (t-r)^{-\frac{\alpha}{2}} dr + C \left(\sup_{u \in [0, T]} \|y(u)\|_{L^p(\Omega; U^s)} + 1 \right) t^{\frac{2H-\alpha}{2}} \\
&\leq C < \infty.
\end{aligned} \tag{3.11}$$

Hence, $\Phi_{y_0} : \Theta_s^p \rightarrow \Theta_s^p$ is a well-defined bounded operator. Let $y(t), z(t) \in \Theta_s^p$ are solutions of the equation (2.1), with (2.1), it can be noted that:

$$\begin{aligned}
&\left\| \Phi_{y_0}(y)(t) - \Phi_{y_0}(z)(t) \right\|_{\Theta_s^p} \\
&= \left\| \mathcal{R}_{\alpha, s}(t)[y_0 - y_0] \right\|
\end{aligned}$$

$$\begin{aligned}
& + \int_0^t \mathcal{R}_{\alpha,s}(t-u) [\mathbf{h}(y(u)) - \mathbf{h}(z(u))] du \\
& + \int_0^t \mathcal{R}_{\alpha,s}(t-u) [\mathbf{g}(y(u)) - \mathbf{g}(z(u))] dB^H(u) \Big\|_{\Theta_s^p} \\
& \leq \int_0^t \|\mathcal{R}_{\alpha,s}(t-u)\| \|\mathbf{h}(y(u)) - \mathbf{h}(z(u))\|_{\Theta_s^p} du \\
& \quad + \left\| \int_0^t \mathcal{R}_{\alpha,s}(t-u) [\mathbf{g}(y(u)) - \mathbf{g}(z(u))] dB^H(u) \right\|_{\Theta_s^p} \\
& \leq C \int_0^t C du \cdot \|y(t) - z(t)\|_{\Theta_s^p} + I_3 \\
& \leq Ct \cdot \|y(t) - z(t)\|_{\Theta_s^p} + I_3.
\end{aligned} \tag{3.12}$$

Following the argument used in the proof of [Lemma 2](#) leads to an estimate for I_3 :

$$\begin{aligned}
I_3 & \leq C \sup_{t \in [0, T]} \left\| \left(\int_0^t \|\Gamma_{H,t}^* \mathcal{R}_{\alpha,s}(t-u) [\mathbf{g}(y(u)) - \mathbf{g}(z(u))]\|_{\mathcal{L}_2^0}^2 du \right)^{\frac{1}{2}} \right\|_{L^p(\Omega; \mathbb{R})} \\
& \quad + C \sup_{t \in [0, T]} \left\| \left(\int_0^t \int_0^t \|\mathbb{D}_s^H \mathcal{R}_{\alpha,s}(t-u) [\mathbf{g}(y(t)) - \mathbf{g}(z(t))]\|_{\mathcal{L}_{2,s}^0}^2 ds du \right)^{\frac{1}{2}} \right\|_{L^p(\Omega; \mathbb{R})} \\
& \leq C \sup_{t \in [0, T]} \|\mathbf{g}(y(t)) - \mathbf{g}(z(t))\|_{L^p(\Omega; U^s)} \left(\int_0^t \|\Gamma_{H,t}^* \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u)\|^2 du \right)^{\frac{1}{2}} \\
& \quad + C \sup_{t \in [0, T]} \|\mathbf{g}(y(t)) - \mathbf{g}(z(t))\|_{L^p(\Omega; U^s)} t^{1-\frac{\alpha}{2}} \\
& \leq Ct^{\frac{2H-\alpha}{2}} \sup_{t \in [0, T]} \|y(t) - z(t)\|_{L^p(\Omega; U^s)} + Ct^{1-\frac{\alpha}{2}} \sup_{t \in [0, T]} \|y(t) - z(t)\|_{L^p(\Omega; U^s)} \\
& = C \left(t^{\frac{2H-\alpha}{2}} + t^{1-\frac{\alpha}{2}} \right) \|y(t) - z(t)\|_{\Theta_s^p}.
\end{aligned} \tag{3.13}$$

Consequently, one arrives at

$$\|\Phi_{y_0}(y)(t) - \Phi_{y_0}(z)(t)\|_{\Theta_s^p} \leq C \left(t + t^{\frac{2H-\alpha}{2}} + t^{1-\frac{\alpha}{2}} \right) \|y(t) - z(t)\|_{\Theta_s^p}.$$

Choosing a sufficiently small time τ and letting $t \in (0, \tau)$ such that

$$C \left(t + t^{\frac{2H-\alpha}{2}} + t^{1-\frac{\alpha}{2}} \right) < 1.$$

Similar to the proof of [Theorem 2](#), it can be shown that $\sup_{t \in [0, \tau]} \|y(t)\|_{L^p(\Omega; U^s)} \leq C$, where C is independent of τ . Together with the extension theorem, [\(2.1\)](#) admits a unique globally mild solution.

□

3.3. Spatial regularity

Theorem 2. Let the assumptions of [Lemma 2](#) and [\(2.16\)](#) be fulfilled. Hence, one can find a constant C satisfying

$$\sup_{t \in [0, T]} \|y(t)\|_{L^p(\Omega; U^s)} \leq C.$$

Proof. According to [\(3.1\)](#):

$$\begin{aligned}
\|y(t)\|_{L^p(\Omega; U^s)} & = \left\| \mathcal{R}_{\alpha,s}(t)y(0) + \int_0^t \mathcal{R}_{\alpha,s}(t-r)\mathbf{h}(y(r)) dr + \int_0^t \mathcal{R}_{\alpha,s}(t-r)\mathbf{g}(y(r)) dB^H(r) \right\|_{L^p(\Omega; U^s)} \\
& \leq \|\mathcal{R}_{\alpha,s}(t)y(0)\|_{L^p(\Omega; U^s)} + \left\| \int_0^t \mathcal{R}_{\alpha,s}(t-r)\mathbf{h}(y(r)) dr \right\|_{L^p(\Omega; U^s)} \\
& \quad + \left\| \int_0^t \mathcal{R}_{\alpha,s}(t-r)\mathbf{g}(y(r)) dB^H(r) \right\|_{L^p(\Omega; U^s)} \\
& = I_1 + I_2 + I_3.
\end{aligned} \tag{3.14}$$

Firstly, using [\(2.15\)](#) to estimate I_1 yields:

$$I_1 = \|\mathcal{R}_{\alpha,s}(t)y(0)\|_{L^p(\Omega; U^s)}$$

$$\begin{aligned}
&\leq \|\mathcal{R}_{\alpha,s}(t)\| \cdot \|y_0\|_{L^p(\Omega;U^s)} \\
&\leq C\|y_0\|_{L^p(\Omega;U^s)} \\
&\leq C.
\end{aligned} \tag{3.15}$$

Next, for I_2 , applying Hölder's inequality and Jensen's inequality, combining (2.16) yields :

$$\begin{aligned}
I_2 &\leq \int_0^t \|\mathcal{R}_{\alpha,s}(t-r)\|_{\mathcal{L}_{2,s}^0} \|\mathbf{h}(y(r))\|_{L^p(\Omega;U^s)} dr \\
&\leq \left[\int_0^t \|\mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-r)\|^2 dr \right]^{\frac{1}{2}} \left[\int_0^t (1 + \|y(r)\|_{L^p(\Omega;U^s)})^2 dr \right]^{\frac{1}{2}} \\
&\leq C \left[\int_0^t \left((t-r)^{-\frac{\alpha}{2}} \right)^2 dr \right]^{\frac{1}{2}} \left(\int_0^t (1 + \|y(r)\|_{L^p(\Omega;U^s)})^2 dr \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t (1 + \|y(r)\|_{L^p(\Omega;U^s)})^2 dr \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t 1 dr + \int_0^t \|y(r)\|_{L^p(\Omega;U^s)}^2 dr \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t 1 dr \right)^{\frac{1}{2}} + C \left(\int_0^t \|y(r)\|_{L^p(\Omega;U^s)}^2 dr \right)^{\frac{1}{2}} \\
&\leq C + C \left(\int_0^t \|y(r)\|_{L^p(\Omega;U^s)}^2 dr \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.16}$$

The estimate for I_3 can be derived in a manner analogous to the proof of Lemma 2, yielding:

$$\begin{aligned}
I_3 &= \left\| \int_0^t \mathcal{R}_{\alpha,s}(t-r) \mathbf{g}(y(r)) d\mathbf{B}^H(r) \right\|_{L^p(\Omega;U^s)} \\
&\leq C \left\| \left(\int_0^t \left\| \Gamma_{H,t}^* \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 (1 + \|y(u)\|_{L^p(\Omega;U^s)}^2) du \right)^{\frac{1}{2}} \right\| \\
&\quad + C \left(\int_0^t \int_0^t \left\| \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 (1 + \|y(u)\|_{L^p(\Omega;U^s)})^2 dudr \right)^{\frac{1}{2}} \\
&\leq C \left\| \int_0^t \left\| \Gamma_{H,t}^* \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 du + \left(\int_0^t \left\| \Gamma_{H,t}^* \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 \|y(u)\|_{L^p(\Omega;U^s)}^2 du \right)^{\frac{1}{2}} \right\| \\
&\quad + C \left(\int_0^t \int_0^t \left\| \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 dudr + \int_0^t \int_0^t \left\| \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 \|y(u)\|_{L^p(\Omega;U^s)}^2 dudr \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t \left\| \Gamma_{H,t}^* \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 du \right)^{\frac{1}{2}} + C \left(\int_0^t \left\| \Gamma_{H,t}^* \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 \|y(u)\|_{L^p(\Omega;U^s)}^2 du \right)^{\frac{1}{2}} \\
&\quad + C \left(\int_0^t \left\| \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 du \cdot t \right)^{\frac{1}{2}} + C \left(\int_0^t \left\| \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 \|y(u)\|_{L^p(\Omega;U^s)}^2 du \cdot t \right)^{\frac{1}{2}} \\
&\leq Ct^{\frac{2H-\alpha}{2}} + C \left(\int_0^t \left\| \Gamma_{H,t}^* \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 \|y(u)\|_{L^p(\Omega;U^s)}^2 du \right)^{\frac{1}{2}} \\
&\quad + C \left(\int_0^t \left\| \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-u) \right\|^2 \|y(u)\|_{L^p(\Omega;U^s)}^2 du \right)^{\frac{1}{2}} t^{\frac{1}{2}}.
\end{aligned} \tag{3.17}$$

In the case of $H \in (0, \frac{1}{2})$:

$$\begin{aligned}
I_3 &\leq Ct^{\frac{2H-\alpha}{2}} + C \left(\int_0^t (t-u)^{-\alpha} \|y(u)\|_{L^p(\Omega;U^s)}^2 du \right)^{\frac{1}{2}} \\
&\quad + C \left(\int_0^t u^{1-2H} (t-u)^{2H-\alpha-1} \|y(u)\|_{L^p(\Omega;U^s)}^2 du \right)^{\frac{1}{2}} \\
&\quad + C \left(\int_0^t u^{-2H+1} (t-u)^{-\alpha} \|y(u)\|_{L^p(\Omega;U^s)}^2 du \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.18}$$

Gathering the above inequality (3.18) with Cauchy's inequality yields:

$$\begin{aligned}
\|y(t)\|_{L^p(\Omega;U^s)}^2 &\leq C + C \int_0^t \|y(u)\|_{L^p(\Omega;U^s)}^2 du + Ct^{2H-\alpha} + Ct \int_0^t (t-u)^{-\alpha} \|y(u)\|_{L^p(\Omega;U^s)}^2 du \\
&\quad + Ct^{2H-1} \int_0^t u^{1-2H} (t-u)^{2H-\alpha-1} \|y(u)\|_{L^p(\Omega;U^s)}^2 du \\
&\quad + C \int_0^t u^{1-2H} (t-u)^{-\alpha} \|y(u)\|_{L^p(\Omega;U^s)}^2 du \\
&\leq C + C \int_0^t (t-u)^{-\alpha} \|y(u)\|^2 du + \int_0^t u^{1-2H} (t-u)^{-\alpha} \|y(u)\|_{L^p(\Omega;U^s)}^2 du.
\end{aligned} \tag{3.19}$$

Since $0 < \alpha < 1$ and $1 - 2H > -\alpha$, via the generalized Gronwall inequality in [5] yields the existence of constants $\mu > 0$ and $\theta \in (0, 1 - \alpha)$ such that:

$$\|y(t)\|_{L^p(\Omega;U^s)}^2 \leq 2Ce^{\theta\mu t} \leq Ce^{\theta\mu t} < L.$$

Hence, the estimate $\sup_{t \in [0, T]} \|y(t)\|_{L^p(\Omega;U^s)} < C$ is obtained.

In the case of $H \in (\frac{1}{2}, 1)$, it should be noted that:

$$\begin{aligned}
\|y(t)\|_{L^p(\Omega;U^s)}^2 &\leq C + C \int_0^t \|y(u)\|_{L^p(\Omega;U^s)}^2 du + Ct^{\frac{2H-\alpha}{2}} \\
&\quad + C \int_0^t u^{1-2H} (t-u)^{-\alpha+2H-1} \|y(u)\|_{L^p(\Omega;U^s)}^2 du \\
&\leq C + C \int_0^t u^{1-2H} (t-u)^{-\alpha+2H-1} \|y(u)\|_{L^p(\Omega;U^s)}^2 du.
\end{aligned} \tag{3.20}$$

Since $0 < 2H - \alpha \leq 1$ and $1 - 2H > \alpha - 2H$, it follows that:

$$\sup_{t \in [0, T]} \|y(t)\|_{L^p(\Omega;U^s)} \leq C.$$

In summary, this establishes the space regularity of the solution:

$$\sup_{t \in [0, T]} \|y(t)\|_{L^p(\Omega;U^s)} \leq C.$$

□

3.4. Temporal regularity

Theorem 3. Let $p \geq 2$, $\alpha < 2H$, $\gamma \in [0, \frac{2H-\alpha}{2})$, suppose (2.16) and (2.17) hold, $y_0 \in L^p(\Omega; U^{s+2\gamma})$. Then the following estimate holds for any $u \in (0, t)$ with some constant C , such that:

$$\|y(t) - y(u)\|_{L^p(\Omega;U^s)} \leq C(t-u)^\gamma.$$

Proof. For any $u \in (0, t)$, using (3.1) to get:

$$\begin{aligned}
\|y(t) - y(u)\|_{L^p(\Omega;U^s)} &= \left\| [\mathcal{R}_{\alpha,s}(t) - \mathcal{R}_{\alpha,s}(u)]y_0 \right. \\
&\quad + \int_0^t \mathcal{R}_{\alpha,s}(t-r)\mathbf{h}(y(r)) dr - \int_0^u \mathcal{R}_{\alpha,s}(u-r)\mathbf{h}(y(r)) dr \\
&\quad + \int_0^t \mathcal{R}_{\alpha,s}(t-r)\mathbf{g}(y(r)) dB^H(r) - \int_0^u \mathcal{R}_{\alpha,s}(u-r)\mathbf{g}(y(r)) dB^H(r) \left. \right\|_{L^p(\Omega;U^s)} \\
&\leq \|[\mathcal{R}_{\alpha,s}(t) - \mathcal{R}_{\alpha,s}(u)]y_0\|_{L^p(\Omega;U^s)} \\
&\quad + \left\| \int_0^u [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)]\mathbf{h}(y(r)) dr \right\|_{L^p(\Omega;U^s)} \\
&\quad + \left\| \int_u^t \mathcal{R}_{\alpha,s}(t-r)\mathbf{h}(y(r)) dr \right\|_{L^p(\Omega;U^s)} \\
&\quad + \left\| \int_0^u [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)]\mathbf{g}(y(r)) dB^H(r) \right\|_{L^p(\Omega;U^s)} \\
&\quad + \left\| \int_u^t \mathcal{R}_{\alpha,s}(t-r)\mathbf{g}(y(r)) dB^H(r) \right\|_{L^p(\Omega;U^s)} \\
&= I_0 + I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.21}$$

Using the Lemma 2.1 from [26], let $\gamma \in [0, 1]$, I_0 can be estimated as follow:

$$\begin{aligned} I_0 &= \|[\mathcal{R}_{\alpha,s}(t) - \mathcal{R}_{\alpha,s}(u)]y_0\|_{L^p(\Omega;U^s)} \leq \|\mathbb{A}^{-\gamma}[\mathcal{R}_{\alpha,s}(t) - \mathcal{R}_{\alpha,s}(u)]\| \|\mathbb{A}^\gamma y_0\|_{L^p(\Omega;U^s)} \\ &\leq \left\| \int_u^t \mathbb{A}^\gamma \mathcal{R}_{\alpha,s}(r) dr \right\| \|y_0\|_{L^p(\Omega;U^{s+2\gamma})} \leq C \int_u^t r^{\alpha\gamma-1} dr \\ &= C(t^{\alpha\gamma} - u^{\alpha\gamma}) \leq C(t-u)^{\alpha\gamma} \leq C(t-u)^\gamma, \end{aligned} \quad (3.22)$$

and on the basis of inequality $|e^{\zeta u} - 1| \leq C|\zeta|^\gamma |u|^\gamma$, $\gamma \in (0, 1)$ and (3.8),

$$\begin{aligned} I_1 &\leq \int_0^u \left\| \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\| \cdot \|\mathbf{h}(y(r))\|_{L^p(\Omega;U^s)} dr \\ &\leq \int_0^u (u-r)^{-\gamma} (t-u)^\gamma (1 + \|y(r)\|_{L^p(\Omega;U^s)}) dr \\ &\leq C \int_0^u (u-r)^{-\gamma} (t-u)^\gamma dr \leq C(t-u)^\gamma. \end{aligned} \quad (3.23)$$

For I_2 , when $H \in (0, 1)$, gathering (2.16) and (2.15) gives:

$$\begin{aligned} I_2 &= \left\| \int_u^t \mathcal{R}_{\alpha,s}(t-r) \mathbf{h}(y(r)) dr \right\|_{L^p(\Omega;U^s)} \leq \int_u^t \left\| \mathbb{A}^{\frac{s}{2}} \mathcal{R}_{\alpha,s}(t-r) \right\| \|\mathbf{h}(y(r))\|_{L^p(\Omega;U^s)} dr \\ &\leq C \int_u^t (t-r)^{-\frac{\alpha}{2}} dr \leq C(t-u)^{1-\frac{\alpha}{2}} \leq C(t-u)^\gamma. \end{aligned} \quad (3.24)$$

Here, $-2\gamma - \alpha > -1$, i.e., $\gamma < \frac{1-\alpha}{2}$.

For I_4 , it's easy to get:

$$\begin{aligned} I_4 &= \left\| \int_u^t \mathcal{R}_{\alpha,s}(t-r) \mathbf{g}(y(r)) dB^H(r) \right\|_{L^p(\Omega;U^s)} \leq C \left(\sup_{u \in [0,T]} \|y(u)\|_{L^p(\Omega;U^s)} + 1 \right) (t-u)^{\frac{2H-\alpha}{2}} \\ &\leq C(t-u)^{\frac{2H-\alpha}{2}} \leq C(t-u)^\gamma. \end{aligned} \quad (3.25)$$

Now, to estimate I_3 :

$$\begin{aligned} I_3 &= \left\| \int_0^u [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \mathbf{g}(y(r)) dB^H(r) \right\|_{L^p(\Omega;U^s)} \\ &= \left\| \int_0^u \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \mathbf{g}(y(r)) dB^H(r) \right\|_{L^p(\Omega;U^s)} \\ &\leq \left\| \left(\int_0^u \left\| \Gamma_{H,u}^* [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \mathbb{A}^{\frac{s}{2}} \mathbf{g}(y(r)) \right\|_{\mathcal{L}_2^0}^2 dr \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;U^s)} \\ &\quad + \left\| \left(\int_0^u \int_0^u \left\| \mathbb{D}_s^H [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \mathbb{A}^{\frac{s}{2}} \mathbf{g}(y(r)) \right\|_{\mathcal{L}_{2,s}^0}^2 dr du \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;U^s)} \\ &\leq \left\| \left(\int_0^u \left\| \Gamma_{H,u}^* \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\| \left\| \mathbf{g}(y(r)) \right\|_{\mathcal{L}_2^0} dr \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;U^s)} + J_2 \\ &\leq C \left\| \left(\int_0^u \left\| \Gamma_{H,u}^* \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\| dr \right)^{\frac{1}{2}} \right\|_{L^p(\Omega;\mathbb{R})} + J_2 = C J_1 + J_2. \end{aligned} \quad (3.26)$$

For J_2 , in the case of $H \in (0, 1)$, combining (2.20) and (2.13) to get:

$$\begin{aligned} J_2 &\leq \left(\int_0^u \int_0^u \left(\left\| \mathbb{D}_v^H \mathbb{A}^{\frac{s}{2}} \mathbf{g}(y(r)) \right\|_{\mathcal{L}_2^0} \left\| [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\| \right)^2 dv dr \right)^{\frac{1}{2}} \\ &\leq C \left(u \int_0^u \left\| [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\|^2 dr \right)^{\frac{1}{2}} \\ &\leq C \left(\int_0^u \left\| \frac{1}{2\pi i} \int_\Gamma e^{(t-u)\zeta} - 1 \left\| e^{(u-r)\zeta} \right\| |\zeta|^{-1} |d\zeta| \right\|^2 dr \right)^{\frac{1}{2}} \\ &\leq C \left(\int_0^u \left\| \int_\Gamma (t-u)^\gamma e^{(u-r)\zeta} |\zeta|^{\gamma-1} |d\zeta| \right\|^2 dr \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\int_0^u [(u-r)^{-\gamma} (t-u)^\gamma]^2 dr \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^u (u-r)^{-2\gamma} dr \right)^{\frac{1}{2}} (t-u)^\gamma \leq C(t-u)^\gamma,
\end{aligned} \tag{3.27}$$

where $-2\gamma > -1$, i.e., $\gamma < \frac{1}{2}$.

For the estimate of J_1 , when $H \in (0, \frac{1}{2})$,

$$\begin{aligned}
J_1^2 &\leq \left\| \int_0^u \Gamma_{H,u}^* \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] dr \right\|_{L^p(\Omega; \mathbb{R})}^2 \\
&\leq C \int_0^u \left\| \mathbf{K}_H(u, r) \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\|^2 dr \\
&\quad + \int_0^u \left\| \int_r^u \mathbb{A}^{\frac{s}{2}} \left\{ [\mathcal{R}_{\alpha,s}(t-v) - \mathcal{R}_{\alpha,s}(u-v)] \right. \right. \\
&\quad \left. \left. - [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\} \frac{\partial \mathbf{K}_H(v, r)}{\partial u} du \right\|^2 dr \\
&= C(J_{1,1} + J_{1,2}).
\end{aligned} \tag{3.28}$$

For $J_{1,1}$, according to (2.5):

$$\begin{aligned}
J_{1,1} &\leq \int_0^u \left\| \left(\frac{u}{r} \right)^{H-\frac{1}{2}} (u-r)^{H-\frac{1}{2}} \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\|^2 dr \\
&\quad + C \int_0^u \left\| r^{-H+\frac{1}{2}} \left(\int_r^u v^{H-\frac{3}{2}} (v-r)^{H-\frac{1}{2}} dv \right) \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\|^2 dr \\
&\leq J_{1,1,1} + J_{1,1,2}.
\end{aligned} \tag{3.29}$$

Next, $J_{1,1,1}$ is estimated by applying the same technique used to derive (3.8), leading to:

$$\begin{aligned}
J_{1,1,1} &\leq \int_0^u \left[(u-r)^{H-\frac{1}{2}} (u-r)^{-\frac{\alpha}{2}-\gamma} (t-u)^\gamma \right]^2 dr \\
&\leq \int_0^u (u-r)^{2H-2\gamma-\alpha-1} dr (t-u)^{2\gamma} \leq C(t-u)^{2\gamma},
\end{aligned} \tag{3.30}$$

where $2H-2\gamma-\alpha-1 > -1$, i.e., $\gamma < \frac{2H-\alpha}{2}$, $H > \frac{\alpha}{2}$.

To estimate $J_{1,1,2}$, use the transformation $v = \frac{r}{\xi}$.

$$\begin{aligned}
J_{1,1,2} &\leq \int_0^u \left[r^{-H+\frac{1}{2}} \left(\int_r^u v^{H-\frac{3}{2}} (v-r)^{H-\frac{1}{2}} dv \right) (t-u)^\gamma (u-r)^{-\gamma-\frac{\alpha}{2}} \right]^2 dr \\
&\leq \int_0^u \left[r^{-H+\frac{1}{2}} (u-r)^{-\gamma-\frac{\alpha}{2}} \left(\int_r^u v^{H-\frac{3}{2}} (v-r)^{H-\frac{1}{2}} dv \right) \right]^2 dr (t-u)^{2\gamma} \\
&\leq \int_0^u r^{-2H+1} (u-r)^{-2\gamma-\alpha} \left[\int_r^u v^{H-\frac{3}{2}} (v-r)^{H-\frac{1}{2}} dv \right]^2 dr (t-u)^{2\gamma} \\
&\leq \int_0^u r^{-2H+1} (u-r)^{-2\gamma-\alpha} \left[\int_{\frac{r}{u}}^1 v^{-2H} (1-v)^{H-\frac{1}{2}} dv \right]^2 dr (t-u)^{2\gamma} \\
&\leq \int_0^u r^{-1+2H} (u-r)^{-2\gamma-\alpha} \left[\int_0^1 \xi^{-2H} (1-\xi)^{H-\frac{1}{2}} d\xi \right]^2 dr (t-u)^{2\gamma} \\
&\leq C(t-u)^{2\gamma}.
\end{aligned} \tag{3.31}$$

where $-2\gamma-\alpha > -1$, i.e., $\gamma < \frac{1-\alpha}{2}$. Hence, $J_{1,1} \leq J_{1,1,1} + J_{1,1,2} \leq C(t-u)^{2\gamma}$.

Applying (2.13) and following a similar argument as in (3.8) yield:

$$\begin{aligned}
&J_{1,2} \\
&= \int_0^u \left\| \int_r^u \mathbb{A}^{\frac{s}{2}} \left\{ [\mathcal{R}_{\alpha,s}(t-v) - \mathcal{R}_{\alpha,s}(u-v)] - [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\} \right. \\
&\quad \left. \cdot \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right\|^2 dr
\end{aligned}$$

$$\begin{aligned}
&= \int_0^u \left\| \int_r^u \left[\frac{1}{2\pi i} \int_{\Gamma} \left([e^{\zeta(t-v)} - e^{\zeta(u-v)}] - [e^{\zeta(t-r)} - e^{\zeta(u-r)}] \right) \mathbb{A}^{\frac{s}{2}} \widetilde{\mathcal{R}}(\zeta) d\zeta \right] \right. \\
&\quad \left. \cdot \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right\|^2 dr \\
&= \int_0^u \left\| \int_r^u \left[\frac{1}{2\pi i} \int_{\Gamma} \left([e^{\zeta(t-u)} - 1] e^{\zeta(u-v)} - [e^{\zeta(t-r)} - 1] e^{\zeta(u-r)} \right) \mathbb{A}^{\frac{s}{2}} \widetilde{\mathcal{R}}(\zeta) d\zeta \right] \right. \\
&\quad \left. \cdot \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right\|^2 dr \\
&= \int_0^u \left\| \int_r^u \left[\frac{1}{2\pi i} \int_{\Gamma} (e^{\zeta(t-u)} - 1) (1 - e^{\zeta(v-r)}) e^{\zeta(u-v)} \mathbb{A}^{\frac{s}{2}} \widetilde{\mathcal{R}}(\zeta) d\zeta \right] \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (u-r)^{H-\frac{3}{2}} dv \right\|^2 dr \\
&\leq \int_0^u \left\| \int_r^u \left[\frac{1}{2\pi i} \int_{\Gamma} |e^{\zeta(t-u)} - 1| |1 - e^{\zeta(v-r)}| |e^{\zeta(u-v)}| \left\| \mathbb{A}^{\frac{s}{2}} \widetilde{\mathcal{R}}(\zeta) \right\| |d\zeta| \right] \right. \\
&\quad \left. \cdot \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right\|^2 dr.
\end{aligned}$$

Using inequality $|e^{\zeta u} - 1| \leq C|\zeta|^{\gamma}|u|^{\gamma}$, $\gamma \in (0, 1)$:

$$\begin{aligned}
&J_{1,2} \\
&\leq \int_0^u \left(\int_r^u (t-u)^{\gamma_1} (v-r)^{\gamma_2} \left[\frac{1}{2\pi i} \int_{\Gamma} |e^{\zeta(u-v)}| |\zeta|^{\gamma_1+\gamma_2+\frac{\alpha}{2}-1} |d\zeta| \right] \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right)^2 dr \\
&\leq C \int_0^u \left(\int_r^u (t-u)^{\gamma_1} (v-r)^{\gamma_2+H-\frac{3}{2}} (u-v)^{-\gamma_1-\gamma_2-\frac{\alpha}{2}} \left(\frac{v}{r} \right)^{H-\frac{1}{2}} dv \right)^2 dr \\
&\leq C \int_0^u \left(\int_r^u (v-r)^{\gamma_2+H-\frac{3}{2}} (u-v)^{-\gamma_1-\gamma_2-\frac{\alpha}{2}} dv \right)^2 dr (t-u)^{2\gamma_1} \\
&\leq C(t-u)^{2\gamma_1}.
\end{aligned} \tag{3.32}$$

There is no loss of generality in setting $\gamma_2 = \frac{1}{2} - H + \varepsilon$, where

$$\begin{cases} \gamma_2 + H - \frac{1}{2} > 0, \\ 1 - \gamma_1 - \gamma_2 - \frac{\alpha}{2} > 0, \\ -2\gamma_1 + 2H - 1 - \alpha > -1. \end{cases} \quad \text{i.e.,} \quad \begin{cases} \gamma_2 > \frac{1}{2} - H, \\ \gamma_1 < 1 - \gamma_2 - \frac{\alpha}{2} < \frac{1}{2} + H - \frac{\alpha}{2}, \\ \gamma_1 < \frac{2H-\alpha}{2}. \end{cases}$$

Therefore,

$$J_1^2 \leq J_{1,1} + J_{1,2} \leq C(t-u)^{2\gamma}, \quad \gamma < \frac{2H-\alpha}{2}. \tag{3.33}$$

In the case of $H \in (\frac{1}{2}, 1)$, combining (2.6) and (2.8) gives:

$$\begin{aligned}
J_1^2 &= \int_0^u \left\| \Gamma_{H,u}^* \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \right\|^2 dr \\
&= \int_0^u \left\| \int_r^u \mathbb{A}^{\frac{s}{2}} [\mathcal{R}_{\alpha,s}(t-r) - \mathcal{R}_{\alpha,s}(u-r)] \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right\|^2 dr \\
&= \int_0^u \left\| \int_r^u \left[\frac{1}{2\pi i} \int_{\Gamma} [e^{\zeta(t-r)} - e^{\zeta(u-r)}] \mathbb{A}^{\frac{s}{2}} \widetilde{\mathcal{R}}(\zeta) d\zeta \right] \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right\|^2 dr \\
&\leq C \int_0^u \left\| \int_r^u \left[\int_{\Gamma} |e^{\zeta(t-u)} - 1| |e^{\zeta(u-r)}| \left\| \mathbb{A}^{\frac{s}{2}} \widetilde{\mathcal{R}}(\zeta) \right\| |d\zeta| \right] \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right\|^2 dr \\
&\leq C \int_0^u \left\| \int_r^u \left[\int_{\Gamma} (t-u)^{\gamma} |e^{\zeta(u-r)}| |\zeta|^{\gamma+\frac{\alpha}{2}-1} |d\zeta| \right] \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right\|^2 dr \\
&\leq C \int_0^u \left(\int_r^u (t-u)^{\gamma} (u-r)^{-\gamma-\frac{\alpha}{2}} \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right)^2 dr
\end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^u (u-r)^{-2\gamma-\alpha} \left(\frac{u}{r}\right)^{2H-1} \left(\int_r^u (v-r)^{H-\frac{3}{2}} dv\right)^2 dr (t-u)^{2\gamma} \\
&\leq C \int_0^u (u-r)^{-2\gamma-\alpha+2H-1} \left(\frac{u}{r}\right)^{2H-1} dr (t-u)^{2\gamma} \\
&\leq C \left(\frac{\Gamma(2-2H)\Gamma(2H-2\gamma-\alpha)}{\Gamma(2-2\gamma-\alpha)}\right)^2 (t-u)^{2\gamma} u^{-2\gamma-\alpha+2H} \leq C (t-u)^{2\gamma}.
\end{aligned} \tag{3.34}$$

In order to make (3.34) meaningful, H, α and γ should be satisfied the following conditions:

$$\begin{cases} -2\gamma - \alpha + 2H > 0, \\ 2 - 2H > 0, \\ 2H - 2\gamma - \alpha > 0, \end{cases} \quad \text{i.e., } \gamma < \frac{2H - \alpha}{2}.$$

Taking (3.21)–(3.25) and (3.33)–(3.34) together, for $H \in (0, 1)$, it follows that:

$$I_3^2 \leq C(t-u)^{2\gamma}.$$

Hence,

$$\|y(t) - y(u)\|_{L^p(\Omega; U^s)} \leq C(t-u)^\gamma, \quad \forall \gamma \in \left[0, \frac{2H - \alpha}{2}\right).$$

□

4. Exponentially asymptotic stability in the p th mean

Firstly, define $\Gamma_\theta, \Gamma_\kappa, \Gamma_{\theta, \kappa}$ as follow:

$$\Gamma_\theta = \{\zeta \in \mathbb{C} : \zeta \neq 0, |\arg \zeta| \leq \theta\},$$

$$\Gamma_{\theta, \kappa} = \{\zeta \in \mathbb{C} : |\zeta| > \kappa, |\arg \zeta| \leq \theta\},$$

$$\Gamma_{\theta, \kappa} = \{\rho e^{\pm i\theta} : \rho \geq \kappa\} \cup \{\kappa e^{i\phi} : |\phi| \leq \theta\},$$

where $\pi/2 < \theta < \pi, \kappa > 0$.

Lemma 4. ([10]) $\mathbb{A}^s$ satisfies the conditions stated in [10]. Then, the following is obtained:

$$\|(\zeta^\alpha + \mathbb{A}^s)^{-1}\|_{L^2 \rightarrow L^2} \leq C |\zeta|^{-\alpha}, \quad \text{for all } s \in (0, 1).$$

Next, the exponential decay estimate for the space-time fractional solution operator $\mathcal{R}_{\alpha, s}$ follows.

Lemma 5. Assume that the assumptions of Lemma 4 hold. Then there exists a constant C such that

$$\|\mathcal{R}_{\alpha, s}(t)\| \leq M e^{-at}.$$

Proof. A contour Γ_0 is constructed such that $\operatorname{Re} \zeta$ is negative and bounded; consequently, the boundary of $\Gamma_{\theta, \kappa}$ is shifted to the left. Let

$$\Gamma_1 : \zeta = -a + r e^{i\theta},$$

$$\Gamma_2 : \zeta = -a + r e^{-i\theta},$$

$$\Gamma_3 : \zeta = -a + R e^{i\phi},$$

where $r \geq R$, and ϕ runs from θ to π , then to $-\pi$, and finally to $-\theta$. Here $R > 0$ is taken sufficiently large, and we choose $a > 0$ with

$$a < \lambda_1^{1/\alpha} \sin(\alpha(\pi - \theta)).$$

Thus a is small enough so that after the translation the new contour Γ_0 lies inside $\Gamma_{\theta, \kappa}$ and satisfies $\operatorname{Re}(-a + R e^{i\phi}) < 0$.

Estimate of $\mathcal{R}_{\alpha, s}(t)$ along the contour Γ_0 :

$$\begin{aligned}
\|\mathcal{R}_{\alpha, s}(t)\|_{L^2 \rightarrow L^2} &\leq \frac{1}{2\pi} \int_{\Gamma_0} |e^{\zeta t}| |\zeta|^{\alpha-1} \|(\zeta^\alpha + \mathbb{A}^s)^{-1}\|_{L^2 \rightarrow L^2} |d\zeta| \\
&\leq \frac{1}{2\pi} \int_{\Gamma_0} |e^{\zeta t}| |\zeta|^{\alpha-1} |\zeta|^{-\alpha} |d\zeta| = \frac{1}{2\pi} \int_{\Gamma_0} |e^{t \operatorname{Re} \zeta}| |e^{it \operatorname{Im} \zeta}| |\zeta|^{\alpha-1} |\zeta|^{-\alpha} |d\zeta| \\
&\leq \int_{\Gamma_0} |e^{t \operatorname{Re} \zeta}| |\zeta|^{\alpha-1} |\zeta|^{-\alpha} |d\zeta| = \int_{\Gamma_0} e^{t \operatorname{Re} \zeta} |\zeta|^{-1} |d\zeta|,
\end{aligned} \tag{4.1}$$

where $|e^{it \operatorname{Im} \zeta}| = |\cos(t \operatorname{Im} \zeta) + i \sin(t \operatorname{Im} \zeta)| = 1$. On Γ_1 and Γ_2 , $\zeta = -a + r e^{i\theta}$. Then $\operatorname{Re} \zeta = -a + r \cos \theta$. Since $\theta \in \left(\frac{\pi}{2}, \pi\right)$, we note that $\cos \theta < 0$. Thus $\operatorname{Re} \zeta = -a - r \cos \theta$, i.e., $e^{t \operatorname{Re} \zeta} = e^{-at} \cdot e^{-rc_\theta t}$.

For $|\zeta|$, when a is small and r is large, we have $|\zeta| \sim r$. Also, $|d\zeta| = dr$. Thus, the integral I_1 along Γ_1 can be estimated as:

$$\begin{aligned}
 I_1 &= \int_{\Gamma_1} e^{t \operatorname{Re} \zeta} |\zeta|^{-1} |d\zeta| \\
 &\sim \int_{\Gamma_1} e^{-at} e^{-tr|\cos \theta|} r^{-1} dr \\
 &\leq 2 \int_{\Gamma_1} e^{-at} e^{-tr|\cos \theta|} r^{-1} dr \\
 &\leq C e^{-at} \int_{\Gamma_1} e^{-tr|\cos \theta|} r^{-1} dr \\
 &= C e^{-at} \int_{R+a \cos \theta}^{\infty} e^{tr \cos \theta} r^{-1} dr.
 \end{aligned} \tag{4.2}$$

Take $u = -tr \cos \theta$. Then $r = -\frac{u}{t \cos \theta}$, $dr = -\frac{du}{t \cos \theta}$.

$$\begin{aligned}
 I_1 &= C e^{-at} \int_{-tR \cos \theta}^{\infty} e^{-u \frac{t \cos \theta}{u}} \cdot \frac{1}{t \cos \theta} du \\
 &= C e^{-at} \int_{-tR \cos \theta}^{\infty} \frac{e^{-u}}{u} du \\
 &\leq C e^{-at}.
 \end{aligned} \tag{4.3}$$

On Γ_2 , a parallel analysis for I_2 leads to the analogous bound $I_2 \leq C e^{-at}$.

On Γ_3 , $\zeta = -a + R e^{i\phi}$, $\phi \in [\theta, \pi] \cup [-\pi, -\theta]$.

$$\operatorname{Re} \zeta = -a + R \cos \phi \leq -a + R \cos \theta.$$

Observe that $R - a \leq |\zeta| \leq a + R$, and $|d\zeta| = |R e^{i\phi} d\phi| = R d\phi$.

$$\begin{aligned}
 I_3 &= \int_{\Gamma_3} \left| e^{\zeta t} \right| |\zeta|^{\alpha-1} \left\| (e^{\zeta^2} + \mathbb{A}^s)^{-1} \right\| |d\zeta| \\
 &\leq \int_{\Gamma_3} e^{-at+Rt \cos \phi} |\zeta|^{\alpha-1} \|\zeta\|^{-\alpha} R d\phi \\
 &\leq \int_{\phi} e^{-at+Rt \cos \phi} \cdot \frac{1}{R-a} \cdot R d\phi \\
 &\leq C e^{-at} \int_{\phi} e^{Rt \cos \phi} d\phi \\
 &= C e^{-at} \left(\int_{\theta}^{\pi} e^{Rt \cos \phi} d\phi + \int_{-\pi}^{-\theta} e^{Rt \cos \phi} d\phi \right) \\
 &= C e^{-at} J_1 + J_2.
 \end{aligned} \tag{4.4}$$

On $[\theta, \pi]$, $e^{tR \cos \phi} \leq e^{tR \cos \theta}$, hence

$$J_1 = \int_{\theta}^{\pi} e^{Rt \cos \phi} d\phi \leq \int_{\theta}^{\pi} e^{Rt \cos \theta} d\phi = (\pi - \theta) e^{Rt \cos \theta} \leq C. \tag{4.5}$$

Similarly, on $[-\pi, -\theta]$, $e^{tR \cos \phi} \leq e^{tR \cos \theta}$, so $J_2 \leq C$.

Therefore,

$$I_3 \leq C e^{-at} (J_1 + J_2) \leq M e^{-at}. \tag{4.6}$$

So,

$$\|\mathcal{R}_{a,s}(t)\| \leq M e^{-at}. \tag{4.7}$$

□

Now a new generalized Gronwall inequality is established.

Lemma 6. Suppose $q, \delta, \beta, m, \sigma, \alpha, \gamma, \xi, b, c, d, e \geq 0$, and let $r : [0, T] \rightarrow \mathbb{R}^+$ be nondecreasing and bounded, $y(t)$ be nonnegative and bounded on $[0, T)$. If the following inequality holds:

$$\begin{aligned}
 y(t) &\leq r(t) + b t^q \int_0^t u^{-\delta} (t-u)^{\beta} y(u) du + c t^m \int_0^t u^{-\sigma} (t-u)^{-\alpha} y(u) du \\
 &\quad + d t^{\xi} \int_0^t u^{-\gamma} y(u) du + e t^{\eta} \int_0^t y(u) du,
 \end{aligned}$$

then it can conclude that exists a positive constant μ such that

$$y(t) \leq 2e^{\theta\mu t} r(t), \quad (4.8)$$

where $t \in [0, T]$. Here, $(0 < \theta < \min\{1, 1 + \sigma, 1 - \gamma, 1 - \alpha, 1 - \delta\})$, such that $\alpha + \theta < 1$ and $\theta + \delta < 1$.

Proof. Invoking Hölder's inequality:

$$\begin{aligned} y(t) &\leq r(t) + br^q \left(\int_0^t t^{-\frac{\delta}{1-\theta}} (t-u)^{-\frac{\beta}{1-\theta}} du \right)^{1-\theta} \left(\int_0^t y^{\frac{1}{\theta}}(u) du \right)^{\theta} \\ &\quad + ct^m \left(\int_0^t t^{\frac{\sigma}{1-\theta}} (t-u)^{-\frac{\alpha}{1-\theta}} du \right)^{1-\theta} \left(\int_0^t y^{\frac{1}{\theta}}(u) du \right)^{\theta} \\ &\quad + dt^{\xi} \left(\int_0^t t^{-\frac{\gamma}{1-\theta}} du \right)^{1-\theta} \left(\int_0^t y^{\frac{1}{\theta}}(u) du \right)^{\theta} + et^{1-\theta+\eta} \left(\int_0^t y^{\frac{1}{\theta}}(u) du \right)^{\theta} \\ &\leq r(t) + \left\{ b \left(\frac{\Gamma\left(\frac{1-\theta-\delta}{1-\theta}\right) \Gamma\left(\frac{1-\theta+\beta}{1-\theta}\right)}{\Gamma\left(2 + \frac{\beta-\delta}{1-\theta}\right)} \right)^{1-\theta} t^{q+1-\theta+\beta-\delta} \right. \\ &\quad + c \left(\frac{\Gamma\left(\frac{\sigma+1-\theta}{1-\theta}\right) \Gamma\left(\frac{1-\theta-\alpha}{1-\theta}\right)}{\Gamma\left(\frac{\sigma-\alpha+1-\theta}{1-\theta}\right)} \right)^{1-\theta} t^{\sigma-\alpha+1-\theta+m} \\ &\quad \left. + d \left(\frac{1-\theta}{1-\theta-\gamma} \right)^{1-\theta} t^{1-\theta-\gamma+\xi} + et^{1-\theta+\eta} \right\} \left(\int_0^t y^{\frac{1}{\theta}}(u) du \right)^{\theta}, \end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function. Since $\alpha + \theta < 1$, $\theta + \delta < 1$, then

$$y^{\frac{1}{\theta}}(t) \leq 2^{\frac{1}{\theta}-1} \left(r^{\frac{1}{\theta}}(t) + C^{\frac{1}{\theta}}(t) \int_0^t y^{\frac{1}{\theta}}(u) du \right),$$

where

$$\begin{aligned} C(t) &= b \left(\frac{\Gamma\left(\frac{1-\theta-\delta}{1-\theta}\right) \Gamma\left(\frac{1-\theta+\beta}{1-\theta}\right)}{\Gamma\left(2 + \frac{\beta-\delta}{1-\theta}\right)} \right)^{1-\theta} t^{q+1-\theta+\beta-\delta} \\ &\quad + c \left(\frac{\Gamma\left(\frac{\sigma+1-\theta}{1-\theta}\right) \Gamma\left(\frac{1-\theta-\alpha}{1-\theta}\right)}{\Gamma\left(\frac{\sigma-\alpha+1-\theta}{1-\theta}\right)} \right)^{1-\theta} t^{\sigma-\alpha+1-\theta+m} \\ &\quad + d \left(\frac{1-\theta}{1-\theta-\gamma} \right)^{1-\theta} t^{1-\theta-\gamma+\beta} \\ &\quad + et^{1-\theta+\eta}. \end{aligned}$$

Let $\mathbf{D}(t) = y^{\frac{1}{\theta}}(t)$, $E(t) = 2^{-\frac{1}{\theta}} r^{\frac{1}{\theta}}(t)$, and

$$\mathbf{h}(t) = 2^{\frac{1}{\theta}-1} C^{\frac{1}{\theta}}(t).$$

Then

$$\mathbf{D}(t) \leq E(t) + \mathbf{h}(t) \int_0^t \mathbf{D}(u) du.$$

Note that $\mathbf{h}(t)$ defines a measurable map from $[0, T]$ to $\mathbb{R}^+$ satisfying

$$\mathbf{h}_T := \int_0^T \mathbf{h}(t) dt < \infty.$$

Thus, using the generalized Gronwall inequality from [5], there exists a constant $\mu > 0$ for which

$$\mathbf{D}(t) \leq 2e^{\mu\theta t} E(t), \quad t \in [0, T].$$

□

With the help of Lemma 6, the exponential asymptotic stability of (2.1) in the p th mean is demonstrated. First, an estimate for $\mathcal{B}_t$ is given.

Lemma 7. Under the hypotheses of Lemma 5, in the case of $H \in (\frac{1}{2}, 1)$, it holds that:

$$\mathbb{E}\|\mathcal{B}_t\|^2 \leq C \left\{ t^{2H-1} \int_0^t u^{-2H+1} (t-u)^{-1+2H} e^{-2a(t-u)} \mathbb{E}\|\mathbf{g}(y(u))\|_{L_2^0}^2 du \right. \\ \left. + \int_0^t \int_0^t e^{-2a(t-r)} \mathbb{E}\|\mathbb{D}_u^H \mathbf{g}(y(r))\|_{L_{2,s}^0}^2 du dr \right\}.$$

In the case of $H \in (0, \frac{1}{2})$, the following is obtained:

$$\mathbb{E}\|\mathcal{B}_t\|^2 \leq C \left\{ t^{-1+2H} \int_0^t r^{1-2H} (t-r)^{2H-1} e^{-2a(t-r)} \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \right. \\ \left. + \int_0^t r^{-1+2H} e^{-2a(t-r)} \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr + \int_0^t (t-r)^{2H+2\gamma-1} e^{-2a(t-r)} \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \right. \\ \left. + \int_0^t \int_0^t e^{-2a(t-r)} \mathbb{E}\|\mathbb{D}_u^H \mathbf{g}(y(r))\|_{L_{2,s}^0}^2 du dr \right\}.$$

Proof. By Lemma 1, it can be seen that:

$$\mathbb{E}\|\mathcal{B}_t\|^2 \leq \mathbb{E} \int_0^t \|\Gamma_{H,t}^* \mathcal{R}_{\alpha,s}(t-r) \mathbf{g}(y(r))\|_{L_2^0}^2 dr + \mathbb{E} \int_0^t \int_0^t \|\mathbb{D}_u^H \mathcal{R}_{\alpha,s}(t-r) \mathbf{g}(y(r))\|_{L_{2,s}^0}^2 du dr \\ \leq \mathbb{E} \int_0^t \|\Gamma_{H,t}^* \mathcal{R}_{\alpha,s}(t-r)\|^2 \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr + \int_0^t \int_0^t \|\mathcal{R}_{\alpha,s}(t-r)\|^2 \mathbb{E}\|\mathbb{D}_u^H \mathbf{g}(y(r))\|_{L_{2,s}^0}^2 du dr \\ \leq I + C \int_0^t \int_0^t e^{-2a(t-r)} \mathbb{E}\|\mathbb{D}_u^H \mathbf{g}(y(r))\|_{L_{2,s}^0}^2 du dr, \quad (4.9)$$

where $I = \int_0^t \|\Gamma_{H,t}^* \mathcal{R}_{\alpha,s}(t-r)\|^2 \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr$.

In the case of $H \in (\frac{1}{2}, 1)$, using (2.6) and (2.8), it holds that:

$$I = \int_0^t \|\Gamma_{H,t}^* \mathcal{R}_{\alpha,s}(t-r)\|^2 \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \\ = \int_0^t \left(\int_r^t \left\| \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} \mathcal{R}_{\alpha,s}(t-r) \right\| dv \right)^2 \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \\ \leq C \int_0^t \left(\int_r^t e^{-a(t-r)} \left(\frac{v}{r} \right)^{H-\frac{1}{2}} (v-r)^{H-\frac{3}{2}} dv \right)^2 \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \\ \leq C \int_0^t \left(\frac{t}{r} \right)^{2H-1} e^{-2a(t-r)} \left(\int_r^t (v-r)^{H-\frac{3}{2}} dv \right)^2 \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \\ \leq C \int_0^t \left(\frac{t}{r} \right)^{2H-1} e^{-2a(t-r)} (t-r)^{2H-1} \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \\ \leq C t^{2H-1} \int_0^t r^{1-2H} (t-r)^{2H-1} e^{-2a(t-r)} \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr. \quad (4.10)$$

Then

$$\mathbb{E}\|\mathcal{B}_t\|^2 \leq C \left\{ t^{2H-1} \int_0^t r^{-2H+1} (t-r)^{-1+2H} e^{-2a(t-r)} \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \right. \\ \left. + \int_0^t \int_0^t e^{-2a(t-r)} \mathbb{E}\|\mathbb{D}_u^H \mathbf{g}(y(r))\|_{L_{2,s}^0}^2 du dr \right\}. \quad (4.11)$$

In the case of $H \in (0, \frac{1}{2})$, according to (2.6), the following result is derived:

$$I = \int_0^t \|\Gamma_{H,t}^* \mathcal{R}_{\alpha,s}(t-r)\|^2 \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \\ \leq \int_0^t \|\mathbf{K}_H(t, r) \mathcal{R}_{\alpha,s}(t-r)\|^2 \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \\ + \int_0^t \left\| \int_r^t [\mathcal{R}_{\alpha,s}(t-v) - \mathcal{R}_{\alpha,s}(t-r)] \frac{\partial \mathcal{R}_H(v, r)}{\partial v} dv \right\|^2 \mathbb{E}\|\mathbf{g}(y(r))\|_{L_2^0}^2 dr \\ = I_{1,1} + I_{1,2}. \quad (4.12)$$

Firstly, $I_{1,1}$ is estimated and divided into two parts according to (2.5):

$$I_{1,1}$$

$$\begin{aligned}
&\leq \int_0^t \left\| \left(\frac{t}{r} \right)^{-\frac{1}{2}+H} (t-r)^{H-\frac{1}{2}} \mathcal{R}_{\alpha,s}(t-r) \right\|^2 \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&\quad + C \int_0^t r^{1-2H} \left(\int_r^t v^{H-\frac{3}{2}} (v-r)^{H-\frac{1}{2}} dv \mathcal{R}_{\alpha,s}(t-r) \right)^2 \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&= J_{1,1} + J_{1,2}.
\end{aligned} \tag{4.13}$$

With Lemma 6, $J_{1,1}$ and $J_{1,2}$ are estimated as follows:

$$\begin{aligned}
J_{1,1} &\leq C \int_0^t \left(\frac{t}{r} \right)^{-1+2H} (t-r)^{2H-1} e^{-2a(t-r)} \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr, \\
J_{1,2} &\leq C \int_0^t r^{1-2H} e^{-2a(t-r)} \left(\int_r^t v^{-\frac{3}{2}+H} (v-r)^{-\frac{1}{2}+H} dv \right)^2 \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr.
\end{aligned}$$

Let $I(v) = \left(\int_r^t v^{H-\frac{3}{2}} (v-r)^{H-\frac{1}{2}} dv \right)^2$, making the substitution $v = r\theta$ yields

$$I(v) \leq C r^{4H-2}.$$

Thus, we obtain

$$J_{1,2} \leq C \int_0^t r^{2H-1} e^{-2a(t-r)} \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr.$$

Hence,

$$\begin{aligned}
I_{1,1} &\leq C \{ t^{2H-1} \int_0^t r^{1-2H} (t-r)^{2H-1} e^{-2a(t-r)} \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&\quad + \int_0^t r^{2H-1} e^{-2a(t-r)} \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \}.
\end{aligned}$$

Through the inequality $\left| \frac{e^{\zeta r}-1}{r^\gamma} \right| < C|\zeta|^\gamma$, $\gamma \in [0, 1]$, and employing the same estimation method as for (3.8) results in:

$$\begin{aligned}
I_{1,2} &= \int_0^t \left\| \int_r^t [\mathcal{R}_{\alpha,s}(t-v) - \mathcal{R}_{\alpha,s}(t-r)] \frac{\partial \mathbf{K}_H(v,r)}{\partial v} dv \right\|^2 \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&= \int_0^t \left\| \int_r^t [\mathcal{R}_{\alpha,s}(t-v) - \mathcal{R}_{\alpha,s}(t-r)] \left(\frac{v}{r} \right)^{-\frac{1}{2}+H} (v-r)^{-\frac{3}{2}+H} dv \right\|^2 \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&= \int_0^t \left\| \int_r^t \frac{1}{2\pi i} \int_{\Gamma_0} [e^{\zeta(t-v)} - e^{\zeta(t-r)}] \widetilde{\mathcal{R}}_{\alpha,s}(\zeta) d\zeta \left(\frac{v}{r} \right)^{-\frac{1}{2}+H} (v-r)^{-\frac{3}{2}+H} dv \right\|^2 \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&\leq C \int_0^t \left(\int_r^t \int_{\Gamma_0} |e^{\zeta(t-r)}| |e^{\zeta(r-v)}| - 1 |\zeta|^{-1} |d\zeta| \left(\frac{v}{r} \right)^{-\frac{1}{2}+H} (v-r)^{-\frac{3}{2}+H} dv \right)^2 \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&\leq C \int_0^t \left(\int_0^t e^{\zeta|t-r|} |\zeta|^{-1+\gamma} |d\zeta| \left(\frac{v}{r} \right)^{-\frac{1}{2}+H} (v-r)^{-\frac{3}{2}+H} dv \right)^2 \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&\leq C \int_0^t e^{-2a(t-r)} \left(\int_r^t \left(\frac{v}{r} \right)^{-\frac{1}{2}+H} (v-r)^{-\frac{3}{2}+H+\gamma} dv \right)^2 \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&\leq C \int_0^t e^{-2a(t-r)} \left(\int_r^t (v-r)^{-\frac{3}{2}+H+\gamma} dv \right)^2 \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&\leq C \int_0^t e^{-2a(t-r)} (t-r)^{-1+2H+2\gamma} \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr.
\end{aligned} \tag{4.14}$$

Here, $H - \frac{1}{2} + \gamma > -1$, i.e., $\gamma > \frac{1}{2} - H$.

Therefore,

$$\begin{aligned}
I &\leq I_{1,1} + I_{1,2} \leq C \{ t^{-1+2H} \int_0^t r^{1-2H} (t-r)^{2H-1} e^{-2a(t-r)} \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&\quad + \int_0^t r^{-1+2H} e^{-2a(t-r)} \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \\
&\quad + \int_0^t (t-r)^{2H+2\gamma-1} e^{-2a(t-r)} \mathbb{E} \|\mathbf{g}(y(r))\|_{\mathcal{L}_2^0}^2 dr \}.
\end{aligned} \tag{4.15}$$

□

Theorem 4. Let $p \geq 2$, $y(t)$ and $z(t)$ be two mild solutions of (2.1) corresponding to initial values y_0 and z_0 , respectively. Assuming the hypotheses of Lemmas 5-6 and condition (2.17), (2.19), (2.21) hold, then for every $t > 0$, one can find a constant μ such that

$$\mathbb{E}\|y(t) - z(t)\|^p \leq C e^{-(pa-\theta\mu)t} \mathbb{E}\|y_0 - z_0\|^p,$$

where $\mu < \frac{pa}{\theta}$.

Proof. When $p = 2$: Let y and z be two solutions of the (2.1), therefore:

$$\begin{aligned} y(t) &= \mathcal{R}_{\alpha,s}(t)y_0 + \int_0^t \mathcal{R}_{\alpha,s}(t-u)\mathbf{h}(y(u)) du + \int_0^t \mathcal{R}_{\alpha,s}(t-u)\mathbf{g}(y(u)) dB^H(u), \\ z(t) &= \mathcal{R}_{\alpha,s}(t)z_0 + \int_0^t \mathcal{R}_{\alpha,s}(t-u)\mathbf{h}(z(u)) du + \int_0^t \mathcal{R}_{\alpha,s}(t-u)\mathbf{g}(z(u)) dB^H(u). \end{aligned}$$

In the case of $H \in (\frac{1}{2}, 1)$, an estimate shows that:

$$\begin{aligned} \mathbb{E}\|y(t) - z(t)\|^2 &\leq C \left\{ e^{-2at} \mathbb{E}\|y_0 - z_0\|^2 \right. \\ &\quad + \int_0^t e^{-2a(t-u)} \mathbb{E}\|y(u) - z(u)\|^2 du \\ &\quad + t^{2H-1} \int_0^t u^{1-2H} (t-u)^{2H-1} e^{-2a(t-u)} \mathbb{E}\|y(u) - z(u)\|^2 du \\ &\quad \left. + \int_0^t \int_0^t e^{-2a(t-u)} \mathbb{E}\|y(u) - z(u)\|^2 dr du \right\} \\ &\leq C \left\{ e^{-2at} \mathbb{E}\|y_0 - z_0\|^2 \right. \\ &\quad + \int_0^t e^{-2a(t-u)} \mathbb{E}\|y(u) - z(u)\|^2 du \\ &\quad + t^{2H-1} \int_0^t u^{1-2H} (t-u)^{2H-1} e^{-2a(t-u)} \mathbb{E}\|y(u) - z(u)\|^2 du \\ &\quad \left. + t \int_0^t e^{-2a(t-u)} \mathbb{E}\|y(u) - z(u)\|^2 du \right\}. \end{aligned} \quad (4.16)$$

Therefore, gathering Lemma 6 and (4.16), one obtains:

$$\frac{\mathbb{E}\|y(t) - z(t)\|^2}{e^{-2at}} \leq 2\mathbb{E}\|y_0 - z_0\|^2 e^{\theta\mu t}.$$

Then

$$\mathbb{E}\|y(t) - z(t)\|^2 \leq C e^{(\theta\mu-2a)t} \mathbb{E}\|y_0 - z_0\|^2. \quad (4.17)$$

When $H \in (0, \frac{1}{2})$, there holds:

$$\begin{aligned} \mathbb{E}\|y(t) - z(t)\|^2 &\leq C \left\{ e^{-2at} \mathbb{E}\|y_0 - z_0\|^2 \right. \\ &\quad + \int_0^t e^{-2a(t-u)} \mathbb{E}\|y(u) - z(u)\|^2 du \\ &\quad + t^{-1+2H} \int_0^t u^{1-2H} (t-u)^{2H-1} e^{-2a(t-u)} \mathbb{E}\|y(u) - z(u)\|^2 du \\ &\quad + \int_0^t u^{2H-1} e^{-2a(t-u)} \mathbb{E}\|y(u) - z(u)\|^2 du \\ &\quad \left. + \int_0^t (t-u)^{2H+2\gamma-1} e^{-2a(t-u)} \mathbb{E}\|y(u) - z(u)\|^2 du \right\}. \end{aligned} \quad (4.18)$$

Therefore, by Lemma 6 and (4.18), the following is obtained:

$$\frac{\mathbb{E}\|y(t) - z(t)\|^2}{e^{-2at}} \leq 2\mathbb{E}\|y_0 - z_0\|^2 e^{\theta\mu t}.$$

Then

$$\mathbb{E}\|y(t) - z(t)\|^2 \leq C e^{(\theta\mu-2a)t} \mathbb{E}\|y_0 - z_0\|^2. \quad (4.19)$$

Now consider the case $p > 2$: By Hölder inequality, it's easy to prove that:

$$\mathbb{E} \|\mathcal{R}_{\alpha,s}(t)y\|^p \leq e^{-pat} \mathbb{E} \|y\|^p, \quad (4.20)$$

and

$$\begin{aligned} \mathbb{E} \left\| \int_0^t \mathcal{R}_{\alpha,s}(t-u) \mathbf{h}(y(u)) du \right\|^p &\leq \left(\int_0^t e^{-2a(t-u)} \mathbb{E} \|\mathbf{h}(y(u))\|^2 du \right)^{\frac{p}{2}} \\ &\leq \left(\int_0^t e^{-pa(t-u)} \mathbb{E} \|\mathbf{h}(y(u))\|^p du \right) \cdot \left(\int_0^t 1^{\frac{p}{p-2}} du \right)^{\frac{p-2}{2}} \\ &\leq t^{\frac{p-2}{2}} \int_0^t e^{-pa(t-u)} \mathbb{E} \|\mathbf{h}(y(u))\|^p du. \end{aligned} \quad (4.21)$$

When $H \in (\frac{1}{2}, 1)$:

$$\begin{aligned} \mathbb{E} \|\mathcal{B}_t\|^p &\leq C \left\{ \mathbb{E} \left(\int_0^t \|\Gamma_{H,t}^* \mathcal{R}_{2,s}(t-u) \mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^2 du \right)^{\frac{p}{2}} \right. \\ &\quad \left. + \mathbb{E} \left(\int_0^t \int_0^t \|\mathbb{D}_u^H \mathcal{R}_{2,s}(t-r) \mathbf{g}(y(r))\|_{\mathcal{L}_{2,s}^0}^2 du dr \right)^{\frac{p}{2}} \right\} \\ &\leq C \left\{ \left(t^{2H-1} \int_0^t u^{1-2H} (t-u)^{2H-1} e^{-2a(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^2 du \right)^{\frac{p}{2}} \right. \\ &\quad \left. + \left(\int_0^t \int_0^t e^{-2a(t-u)} \mathbb{E} \|\mathbb{D}_u^H \mathbf{g}(y(r))\|_{\mathcal{L}_{2,s}^0}^2 dr du \right)^{\frac{p}{2}} \right\} \leq C(I_1 + I_2). \end{aligned}$$

By the Hölder inequality:

$$\begin{aligned} I_1 &\leq t^{2H-1} \left(\int_0^t 1^{\frac{p}{p-2}} du \right)^{\frac{-2+p}{2}} \left(\int_0^t u^{\frac{p(-2H+1)}{2}} (t-u)^{\frac{p(2H-1)}{2}} e^{-pa(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^p du \right) \\ &= t^{2H+\frac{p}{2}-2} \int_0^t u^{\frac{p(-2H+1)}{2}} (t-u)^{\frac{p(2H-1)}{2}} e^{-pa(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^p du, \\ I_2 &\leq t^{\frac{p}{2}-1} \int_0^t \int_0^t e^{-pa(t-u)} \mathbb{E} \|\mathbb{D}_u^H \mathbf{g}(y(r))\|_{\mathcal{L}_{2,s}^0}^p dr du. \end{aligned} \quad (4.22)$$

Therefore,

$$\begin{aligned} \mathbb{E} \|\mathcal{B}_t\|^p &\leq C \left\{ t^{2H+\frac{p}{2}-2} \int_0^t u^{\frac{p(-2H+1)}{2}} (t-u)^{\frac{p(2H-1)}{2}} e^{-pa(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^p du \right. \\ &\quad \left. + t^{\frac{p}{2}-1} \int_0^t \int_0^t e^{-pa(t-u)} \mathbb{E} \|\mathbb{D}_u^H \mathbf{g}(y(r))\|_{\mathcal{L}_{2,s}^0}^p dr du \right\}. \end{aligned} \quad (4.23)$$

By (4.20), (4.21), (4.23):

$$\begin{aligned} \mathbb{E} \|y(t) - z(t)\|^p &\leq 3^{p-1} M^p \left\{ e^{-pat} \mathbb{E} \|y_0 - z_0\|^p + t^{\frac{-2+p}{2}} \int_0^t e^{-pa(t-u)} \mathbb{E} \|y(u) - z(u)\|^p du \right. \\ &\quad \left. + t^{2H+\frac{p}{2}-2} \int_0^t u^{\frac{(1-2H)p}{2}} (t-u)^{\frac{(2H-1)p}{2}} e^{-pa(t-u)} \mathbb{E} \|y(u) - z(u)\|^p du \right. \\ &\quad \left. + t^{\frac{p}{2}} \int_0^t e^{-pa(t-u)} \mathbb{E} \|y(u) - z(u)\|^p du \right\}. \end{aligned} \quad (4.24)$$

Hence, from Lemma 6, we obtain the existence of a positive constant μ with the property that

$$\mathbb{E} \|y(t) - z(t)\|^p \leq C e^{-(pa-\theta\mu)t} \mathbb{E} \|y_0 - z_0\|^p. \quad (4.25)$$

When $H \in (0, \frac{1}{2})$, it holds:

$$\begin{aligned} &\mathbb{E} \|\mathcal{B}_t\|^p \\ &\leq C \left\{ \mathbb{E} \left(\int_0^t \|\Gamma_{H,t}^* \mathcal{R}_{\alpha,s}(t-u) \mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^2 du \right)^{\frac{p}{2}} + \mathbb{E} \left(\int_0^t \int_0^t \|\mathbb{D}_u^H \mathcal{R}_{\alpha,s}(t-r) \mathbf{g}(y(r))\|_{\mathcal{L}_{2,s}^0}^2 du dr \right)^{\frac{p}{2}} \right\} \\ &\leq C \left\{ \left(t^{2H-1} \int_0^t u^{1-2H} (t-u)^{2H-1} e^{-2a(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^2 du \right)^{\frac{p}{2}} \right. \end{aligned}$$

$$\begin{aligned}
& + \int_0^t u^{2H-1} e^{-2a(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^2 du + \int_0^t (t-u)^{2H+2\gamma-1} e^{-2a(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^2 du \Big)^{\frac{p}{2}} \\
& + \left(\int_0^t \int_0^t e^{-2a(t-u)} \mathbb{E} \|\mathbb{D}_u^H \mathbf{g}(y(r))\|_{\mathcal{L}_{2,s}^0}^2 dudr \right)^{\frac{p}{2}} \Big\} \\
& \leq C \left\{ \left(t^{2H-1} \int_0^t u^{1-2H} (t-u)^{2H-1} e^{-2a(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^2 du \right)^{\frac{p}{2}} \right. \\
& \quad + \left(\int_0^t u^{2H-1} e^{-2a(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^2 du \right)^{\frac{p}{2}} \\
& \quad \left. + \left(\int_0^t (t-u)^{2H+2\gamma-1} e^{-2a(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^2 du \right)^{\frac{p}{2}} + J_4 \right\} := J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

By Hölder inequality and (4.22),

$$J_1 \leq t^{2H+\frac{p}{2}-2} \left(\int_0^t u^{\frac{(1-2H)p}{2}} (t-u)^{\frac{(-1+2H)p}{2}} e^{-pa(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^p du \right), \quad (4.26)$$

$$J_2 \leq t^{\frac{p}{2}-1} \int_0^t u^{(-1+2H)p} e^{-pa(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^p du, \quad (4.27)$$

$$J_3 \leq \int_0^t (t-u)^{(H+\gamma)p-1} e^{-pa(t-u)} \mathbb{E} \|\mathbf{g}(y(u))\|_{\mathcal{L}_2^0}^p du. \quad (4.28)$$

$$J_4 \leq t^{\frac{p}{2}-1} \int_0^t \int_0^t e^{-pa(t-u)} \mathbb{E} \|\mathbb{D}_u^H \mathbf{g}(y(r))\|_{\mathcal{L}_{2,s}^0}^p dr du. \quad (4.29)$$

Therefore, by (4.20), (4.21) and (4.26)–(4.29),

$$\begin{aligned}
\mathbb{E} \|y(t) - z(t)\|^p & \leq 3^{p-1} M^p \left\{ e^{-pat} \mathbb{E} \|y_0 - z_0\|^p + t^{\frac{p-2}{p}} \int_0^t e^{-pa(t-u)} \mathbb{E} \|y(u) - z(u)\|^p du \right. \\
& \quad + t^{2H+\frac{p}{2}-2} \int_0^t u^{\frac{(1-2H)p}{2}} (t-u)^{\frac{(p-2H)p}{2}} e^{-pa(t-u)} \mathbb{E} \|y(u) - z(u)\|^p du \\
& \quad + t^{\frac{p}{2}-1} \int_0^t u^{\frac{(-1+2H)p}{2}} e^{-pa(t-u)} \mathbb{E} \|y(u) - z(u)\|^p du \\
& \quad + t^{(H+\gamma)p-1} \int_0^t e^{-pa(t-u)} \mathbb{E} \|y(u) - z(u)\|^p du \\
& \quad \left. + t^{\frac{p}{2}} \int_0^t e^{-pa(t-u)} \mathbb{E} \|y(u) - z(u)\|^p du \right\}.
\end{aligned}$$

Therefore, it follows from Lemma 6, there exists a constant $\mu > 0$ such that

$$\mathbb{E} \|y(t) - z(t)\|^p \leq C e^{-(pa-\theta\mu)t} \mathbb{E} \|y_0 - z_0\|^p. \quad (4.30)$$

□

5. Numerical experiment

This section presents a numerical experiment based on the stochastic space-time fractional diffusion Eq. (2.1) driven by multiplicative fractional Brownian motion. The experiment is designed to verify the stability analysis by examining the exponential decay of the difference between two solutions with perturbed initial conditions. Consider the following example:

$$\begin{cases} dy(t) + {}_0\partial_t^{1-\alpha} \mathbb{A}^s y(t) dt = \mathbf{h}(y(t)) dt + \mathbf{g}(y(t)) dB^H(t), & t \in (0, T], \\ y(0) = y_0 = \sum_{k=1}^{N_x} z_k e^{-0.5k} \phi_k, & z_k \sim \mathcal{N}(0, 1). \end{cases}$$

Here, the fractional derivative of Riemann-Liouville is denoted by ${}_0\partial_t^{1-\alpha}$, and let $\beta = 1 - \alpha$. The operator $\mathbb{A} = -\partial_{xx}$ has Dirichlet boundary conditions, and $B^H(t)$ is a one-dimensional fBm. The functions $\mathbf{h} = -0.2y - 0.05y^3$ and $\mathbf{g} = 0.2y$ meet the assumptions listed in Subsections (2.16)–(2.21).

The numerical simulation employs a semi-implicit scheme with enhanced physical fidelity. The Riemann-Liouville fractional derivative is discretized using the Grünwald-Letnikov (GL) formula:

$${}_0\partial_t^{1-\alpha} u(t_n) \approx \frac{1}{\Delta t^\beta \Gamma(1-\beta)} \sum_{j=0}^n \omega_j^{(\beta)} u(t_{n-j}), \quad (5.1)$$

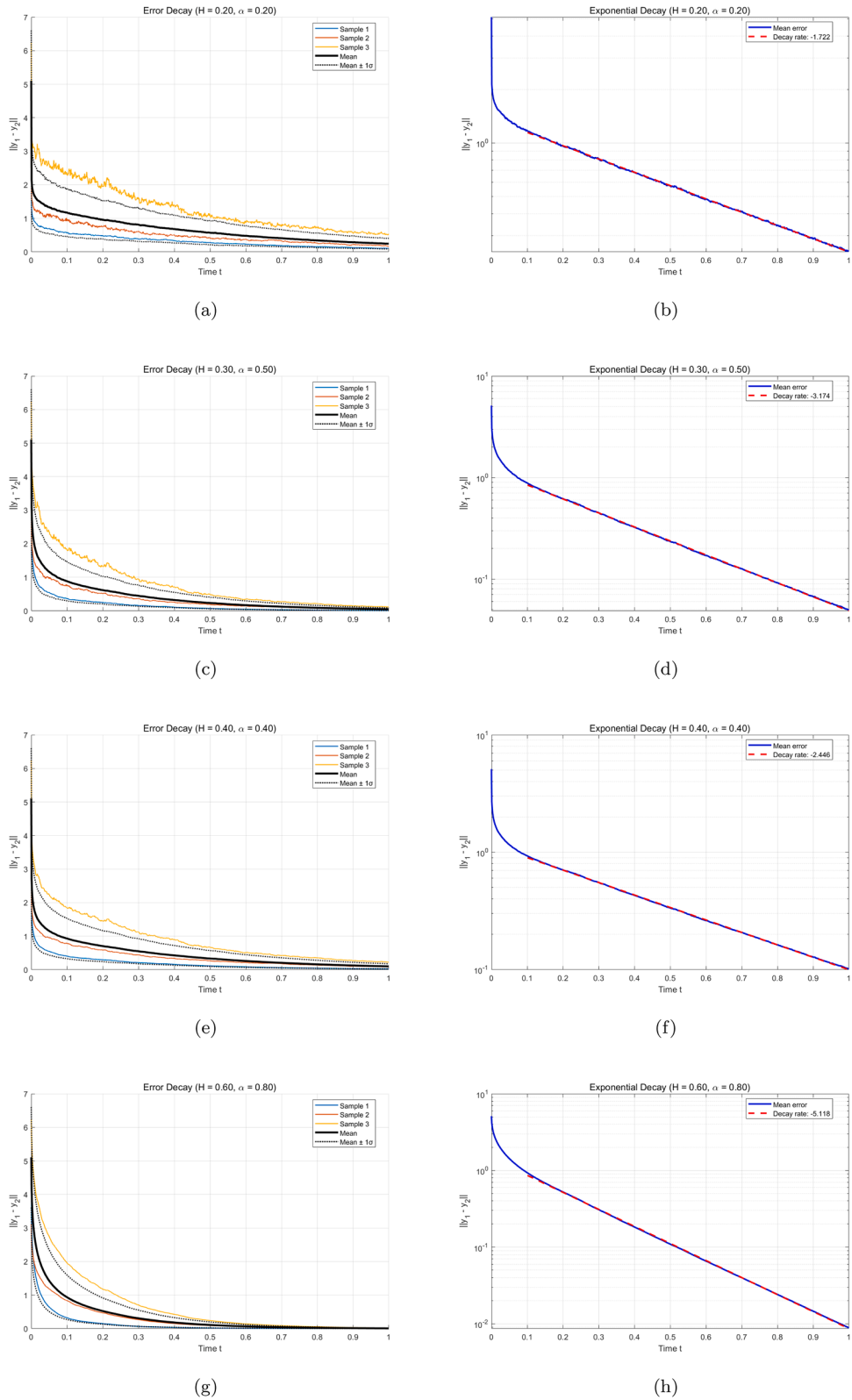


Fig. 1. Solution behaviors for different Hurst parameters $H \in \{0.2, 0.3, 0.4, 0.6\}$ and corresponding $\alpha \in \{0.2, 0.5, 0.4, 0.8\}$.

where $\beta = 1 - \alpha$ and the weights $\omega_j^{(\beta)}$ are computed via the recurrence $\omega_0^{(\beta)} = 1$, $\omega_j^{(\beta)} = \left(1 - \frac{\beta+1}{j}\right)\omega_{j-1}^{(\beta)}$. The fBm increments are generated by exact covariance matrix factorization (Cholesky decomposition of the kernel $R_H(t, s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H})$), ensuring the correct correlation structure.

Two sets of initial spectral coefficients are constructed. The first set consists of standard random coefficients, while the second set is obtained by adding a small perturbation of amplitude 0.1 to the first one, so that the two initial conditions have a non-zero but small difference. For each initial condition, $M = 50$ independent Monte Carlo trajectories are computed using the same fixed noise path. To verify the exponential asymptotic stability stated in [Theorem 4](#), we set $p = 2$. Let $y_1^{(m)}(t)$ and $y_2^{(m)}(t)$ denote the solutions from the two initial data in the m th sample, and define the sample-averaged error

$$\bar{e}(t) = \frac{1}{M} \sum_{m=1}^M \|y_1^{(m)}(t) - y_2^{(m)}(t)\|,$$

where the L^2 -norm is approximated by its discrete analogue $\|y\|^2 = \sum_{k=1}^{N_x} |\hat{y}_k|^2$.

Spatial order $s = 0.8$, final time $T = 1.0$, time step $\Delta t = 0.001$ ($N_t = 1000$), spatial modes $N_x = 16$, and $M = 50$ Monte Carlo samples. Each sample uses an independent random initial condition y_0 with exponentially decaying spectral coefficients.

[Fig. 1](#) displays the evolution of $\bar{e}(t)$ on a semi-logarithmic scale for several combinations of H and α (all satisfying $\alpha < 2H$). After a short initial transient, $\log \bar{e}(t)$ exhibits a clear linear decrease in all cases, indicating an exponential decay of the solution difference. The decay rates obtained by linear fitting for $t > 0.1$ are consistent in magnitude with the theoretical prediction. Taking subfigure (d) in [Fig. 1](#) as an example, the fitted decay rate is approximately -3.174 , which indicates that there exist parameters a , θ , and μ such that $-(2a - \theta\mu) = -3.174$, in agreement with the theoretical exponential decay rate. This exponential decay behavior is in qualitative agreement with [Theorem 4](#) and provides numerical support for the theoretical result.

6. Conclusion

This paper has provided a thorough investigation of a stochastic space-time fractional diffusion equation excited by multiplicative fractional Brownian motion. The key findings fall into three main categories. First, the well-posedness of the problem is demonstrated. Specifically, existence and uniqueness of a mild solution are proved under standard Lipschitz and linear growth conditions on the coefficients. A critical ingredient is the development of precise estimates for the stochastic convolution term involving fractional noise, which hold for the entire Hurst index range $H \in (0, 1)$, including the difficult rough case $H < \frac{1}{2}$. Second, a detailed regularity analysis of the solution is carried out, yielding both spatial and temporal regularity estimates. These estimates serve as the basis for the subsequent stability study. The central theoretical achievement is the establishment of exponential asymptotic stability of the solution in the p th moment. The main difficulties stem from the absence of a semigroup property for the solution operator and the low regularity of the noise when $H \in (0, \frac{1}{2})$. These obstacles are successfully addressed by establishing a novel exponential decay estimate for the fractional solution operator and a tailored generalized Gronwall-type inequality. Combined with stochastic calculus techniques, these analytical tools demonstrate the system's robustness to stochastic perturbations in the long-time sense. The theoretical findings were corroborated by numerical experiments, which illustrated the solution dynamics and their convergence to stability under various parameter settings. Future work may focus on extending these results to models with more general nonlinearities, delays, or driven by other types of noise such as Lévy processes.

CRedit authorship contribution statement

Shixin Yin: Writing – original draft, Validation, Methodology, Formal analysis, Conceptualization; **Ruikun Zhao:** Writing – original draft, Validation, Methodology, Formal analysis, Conceptualization; **Xiaohang Guo:** Writing – original draft, Validation, Methodology, Formal analysis, Conceptualization; **Maosen Zhang:** Writing – original draft, Validation, Methodology, Formal analysis, Conceptualization; **Dongyu Zhan:** Writing – original draft, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization; **Hongtao Fan:** Writing – review & editing, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization; **Yajing Li:** Writing – review & editing, Supervision, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The author declare that they have no conflicts of interest to this work.

Data availability

No data was used for the research described in the article.

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